摘　要

本文利用 Hirota 方法，双线性 Bäcklund变换，Darboux 变换与 Wronskian 技巧对一些非等谱孤子方程与具自溶源 mKP 方程的精确解进行了研究.

在第一、三章中利用 Hirota 方法、双线性 Bäcklund与 Wronskian 技巧分别对非等谱变系数 KdV 、 KP 方程进行了研究. 在第二章中通过 Darboux 变换与双线性 Bäcklund变换方法对非等谱 KP 方程进行了求解. 通过以上对非等谱变系数孤子方程的研究我们可以发现非等谱与等谱有着一定的区别，其特点主要表现在：在等谱孤子方程中振幅随着时间的变化是不变的，而对于非等谱孤子方程是不成立的，在非等谱中振幅是随着时间的变化而变化的；在等谱孤子方程中一般 Darboux 变换与双线性 Bäcklund变换都是自变换，也就是由方程的已知解求出新解，再以所求得的新解作为已知解，求出更新的解，周而复始，在非等谱方程中这两种变换往往是非自变换，即由一个非等谱方程的解不能得到自身的新解，而是得到另一个非等谱方程的解；利用 Hirota 方法得到的解与 Wronskian 技巧得到解在恢复等谱孤子方程的解时是一致的，而对于非等谱方程是不成立的.

第四章主要对具自溶源 mKP 方程进行了研究. 首先由 mKP 系统的线性问题出发，推导出具自溶源 mKP 方程；然后通过一定的变换，具自溶源 mKP 方程可以写成双线性的形式，利用 Hirota 方法不仅可以得到单孤子、双孤子与三孤子解的表达形式，而且可以猜测出 N 孤子解的表达式. 由于在具自溶源 mKP 方程的时间发展式中多出了一个非线性项的表达式，所以在证明 Wronskian 形式的解时就不能按照一般 Wronskian 形式解的证明过程来证明，所以我们提出了一些新的行列式的性质以及证明技巧证明了具自溶源 mKP 方程具有 Wronskian 形式的解. 对于具自溶源 mKP 方程我们得到两种形式的解，其中 Hirota 方法得到的 N 孤子解是猜测出的，而 Wronskian 形式的解仅仅进行了验证，最后我们证明了两种解在恢复孤子方程的解时是一致的.

关键字：Hirota 方法，Wronskian 技巧，双线性 Bäcklund变换，Darboux 变换，非等谱孤子方程，具自溶源 mKP 方程.

中图分类号：

Abstract

In this paper, we consider the solutions of some nonisospectral soliton equations and the mKP equation with self-consistent sources by Hirota method, bilinear Bäcklund transformation, Darboux transformation and Wronskian technique.

In Chap 1 and 3, we consider the nonisospectral and variable-coefficient KdV and KP equations by Hirota method, bilinear Bäcklund transfromation and Wronskian technique. In Chap 2, we study the nonisospectral KP equation by Darboux and bilinear Bäcklund transformation. We also analyze nonisospectral characteristics of the obtain solutions. Solutions in the Hirota's form and in Wronskian form are different both formally and essentially. These two kinds of solutions are not same for recovering the N soliton solutions from the transformations. To the nonisospectral soliton equations, the obtain solutions travel with time-varying shape and speed. It is worthwhile to mention that Darboux transformation and bilinear Bäcklund transformation are auto-Bäcklund transformations for the isospectral soliton equations, but these do not true for the nonisospectral solitons equation. As a matter of fact, they transform one nonisospectral soliton equation to another.

In Chap 4, from the linear problem of mKP equation, we can derive the mKP equation with self-consistent sources. On the other hand, we also hope to find the multi-soliton solutions of the mKP equation with self-consistent sources through Hirota method and Wronskian technique. These two direct methods both depend on the bilinear forms of the evolution equations. We first present a set of dependent variable transformations to write out the bilinear form of the mKP equation with self-consistent sources by which we can derive one-, two-, even three-soliton solutions successively through the standard Hirota's approach. These results can help us to find out the time evolution easily and conjecture a general formula which denotes N-soliton solution but is only conjectured and not verified. Next, with the help of the message on the time evolution obtained by means of Hirota method, we can construct a Wronskian and try to verify it to satisfy the related bilinear equations. Since there is a nonlinear term (led to by the concerned source) in the time evolution, we have to develop some novel determinantal identities and employ some special treatments which are different from the known standard Wronskian technique so that we can finish the Wronskian verifications. Finally, we present a process to show that the solutions of the bilinear equations obtained through the above two direct methods are the same for recovering the solutions of mKP equation with self-consistent sources from the

original dependent variable transformations.

Keywords: Hirota method, Wronskian technique, bilinear Bäcklund transformation, Darboux transformation, nonisospectral soliton equations, mKP equation with self-consistent sources.

Classification Code:

Preface

The soliton theory is an important branch of applied mathematica and mathematical physics. As early as 1844, the study of the solitary wave have been begun. But not until the concept of " soliton " was proposed by Keuskal and Zabusky in the middle of sixties in the present century, was the related research developed rapidly. The theory of solitons is attractive and exciting. It bring together many branches of mathematics, some of which touch on deep ideas, especially in the field of nonlinear mathematics.

In many mathematical subjects of soliton theory, it is an important thing to study the solutions of the nonlinear evolution equations. Over the years, a variety of methods for finding explicit solutions of partial differential equations have been developed. The discovery of the inverse scattering transform (IST) for the KdV equation was a big breakthrough in the analysis of nonlinear evolution equation[43]. Many soliton equations have been revealed to be exactly solvable by this method. Besides the IST, there are some famous direct approaches. Such as, Darboux transformations[15,17], Hirota method[10], and Wronskian technique[12] etc.

In 1971, Hirota[10] first proposed the formal perturbation technique, called the Hirota method later, to get N-soliton solutions of the KdV equation. The soliton solutions can be presented by Wronskian was first proposed by Satsuma[44], but the solutions were not based on the bilinear equation. The Wronskian technique[12] was developed by Freeman and Nimmo for directly verifying solutions to bilinear equation. These two direct methods both depend on the bilinear forms of the evolution equations. Hirota method provides a remarkably simpler technique for obtaining the N-soliton solutions in the form of an Nth-order polynomial in N exponentials. Wronskian technique provides an alternative formulation of the N-soliton solutions, in terms of some function of the Wronski determinant of N functions, which allows verification of the solutions by direct substitution because differentiation of a Wronskian is easy and its derivatives take similar compact forms. To the Hirota method, the basic thoughts of our obtaining the exact N-soliton solutions are as follows. We first present a set of dependent variable transformations to write out the bilinear form of the evolution equation by which we can derive one-, two-, even three-soliton solutions successively. These results can help us to conjecture a general formula which denotes N-soliton solution but is only conjectured and not verified. Next, with the help of the message on the time evolution obtained by means of Hirota method, we can construct a Wronskian and try to verify it to satisfy the related bilinear equations. The Bäcklund transformation (BT) is another direct method for deriving solutions from a known solution of the concerned equation. However, solving the set of partial-differential equations often restricts it to be further used. In 1974, Hirota proposed a kind of BT in bilinear form[11], by which it is easy to find multisoliton solutions. In this method, the linear problem of the evolution equation can been written the bilinear form by a set of transformation. From the bilinear equation, it is easy to derive the soliton solutions. In this paper, similar to the isospectral soliton equation, through a set of transformation we write the nonisospectral KdV and KP equations into the bilinear form. Based on the bilinear form, we can obtain the solutions by Hirota method and Wronskian technique. In chapter 4, the soliton solutions of the mKP equation with self-consistent sources are derived by the Hirota method and Wronskian technique.

Soliton solutions with self-consistent sources are important models in many fields of the physics, such as hydrodynamics, solid-state physics, plasma physics, etc. For example, the nonlinear Schrödinger equation with self-consistent sources represent the nonlinear interaction of an electrostatic high-frequency wave with ion acoustic wave in a two component homogeneous plasma. The KdV equation with self-consistent sources describes the interaction of long and short capillary-gravity waves. The KP equation with self-consistent sources(KPESCS) describes the interaction of long wave with a short-wave packet propagating on the x, y plane at an angle to each other. Until now, much development has been made in the study of soliton equation with self-consistent soources. For example, in (1+1)-Schrödinger, AKNS and Kaup-Newell hierarchies with self-consistent sources were solved by the inverse scattering method. Also a type of generalized binary Darboux transformations with arbitrary functions in time t for some (1+1)-dimensional soliton equation with self-consistent sources, which offer a nonauto-Bäcklund transformation between two soliton equation with self-consistent sources with different degrees of sources, have been constructed and can be used to obtain N-soliton, positon and negaton solution. In (2+1)-dimensional case, some results to the soliton equation with self-consistent sources have been obtained. The soliton solution of the KPESCS was fist found by Mel'nikov[22,23]. However, since the explicit time part of the Lax represent of the KPESCS by inverse scattering transformation was quite complicated[22,23]. In Ref[36], in the framework of Sato theory and by treating the constrained KP hierarchy as the stationary one of the KP hierarchy with self-consistent sources, the Lax representation of the KPESCS were naturally gotten. And the generalized binary Darboux transformation for the KP equation with self-consistent sources was constructed. In Ref[14], we obtain the KPESCS through the linear problem of KP system, the mutisoliton solutions of the KPESCS are derived by the Hirota method and Wronskian technique. In chapter 4, we develop the idea present in Ref[14] to study the mKP equation with self-consistent(mKPESCS). From the linear problem of mKP equation, we get the mKPESCS. Through a set of dependent variable transformations, the bilinear form of the mKPESCS are obtained. By the bilinear equation, we can get the soliton solutions by Hirota method and Wronskian technique.

Recently there has been considerable interest in the study of variable-coefficient generalizations of the soliton equations. The need for studying them is due to the fact that the physical situations in which equations with constant coefficients arise tend to be highly idealized so that equations with variable coefficients and nonisospectral parameters may provide more realistic model, for example, in the propagation of (small-amplitude) surface waves in straits or large channels vorticity. In Ref[4], Chan and Li describe some extensions of the inverse scattering in solving a nonisospectral and variable coefficient KdV equation with time varying nonvanishing boundary condition, obtained some nonpropagating soliton solutions and demonstrate their behavior(the coefficients of the evolution equations are time varying,the scattering problem is nonisospectral and the time varying boundary condition is nonvanishing). In 1992, Chan *et al*[5] obtained the n-soliton solutions for a nonisospectral variable-coefficient KP equation by the dressing method and studied in depth the two-soliton case by appropriately decomposing them into individual solitons in order to examine their interactions. In Chapter 1-3, we study the nonisospectral KdV and KP equations by Hirtoa method, Bäcklund transformation and Wronskian technique. We also analyze nonisospectral characteristics of the obtain solutions. Solutions in the Hirtoa's form and in Wronskian form are different both formally and essentially. These two kinds of solutions are not same for recovering the N soliton solutions from the transformations. To the nonisospectral soliton equations, the obtain solutions travel with time-varying shape and speed. It is worthwhile to mention that Darboux transformation and bilinear Bäcklund transfromation are auto-Bäcklund transformations for the isospectral soliton equations, but these do not true for the nonisospectral solitons equation. As a matter of fact, they transform one nonisospectral soliton equation to another.

Chapter 1
Exact Solutions for a Nonisospectral and Variable-coefficient KdV Equation

The bilinear form for a nonisospectral and variable-coefficient KdV equation is obtained and some exact soliton solutions are derived through Hirota method and Wronskian technique. We also derive the bilinear Bäcklund transformation from its Lax pairs and find solutions with the help of the obtained bilinear Bäcklund transformation.

1.1 Introduction

The physical situation in which many integrable equations with constant coefficients arise tend to be highly idealized. Therefore, equations with variable coefficients and nonisospectral eigenparameters may provide realistic models in many physical situation. Thus, recently there has been much interest in study of the variable coefficients generalizations of completely integrable nonlinear evolution equations[1-9]. For the variable coefficient KdV (vcKdV) equation

$$u_t + h_1(6uu_x + u_{xxx}) + 4h_2u_x - h_3(2u + xu_x) = 0, \tag{1.1.1}$$

where $h_1 = h_1(t), h_2 = h_2(t)$ and $h_3 = h_3(t)$ are all arbitrary function of t. The initial value problem of eq.(1.1.1) was sovled via the inverse scattering method by Chan and Li[4]. Lou *et al.*[7] studied its infinite converse law. Zhang *et al.*[8] discussed its symmetries. Fan *et al.*[9] has obtained the Bäcklund transformation(BT) by the homogeneous balance method.

The Hirota method[10], BT[11] and Wronskian technique[12] are three efficient direct ways to find soliton solutions for nonlinear equations. Recently, Zhang *et al.*[13] study the soliton for the KdV equation with loss and non-uniformity terms by use of Hirota method and Wronskian technique. In this paper, we would like to consider the vcKdV equation through above three methods. The bilinear form of the vcKdV equation is given and one-, two-soliton solutions are obtained through the standard Hirota method. A general formula which denotes higher order solutions is also given. In a way similar to the isospectral equation, from the Lax pairs we can derive the bilinear BT for the vcKdV equation by the variable transformations. We also obtain the solution in Wronskian form. The methods used here can be applied to other nonisospectral soliton equations.

The paper is organized as follows. In Sec.2, we solve the vcKdV equation by the Hirota method. In Sec.3 solution in Wronskian form is proven. In Sec.4, the soliton solutions for the vcKdV equation are obtained by bilinear BT.

1.2 Bilinear form and Hirota method

With the help of the dependent variable transformation

$$u = 2(\ln f)_{xx}, \tag{1.2.1}$$

eq. (1.1.1) can be transformed into the bilinear form

$$D_x D_t f\cdot f + h_1 D_x^4 f\cdot f + 4h_2 D_x^2 f\cdot f - xh_3 D_x^2 f\cdot f - 2h_3 f f_x = 0, \tag{1.2.2}$$

where D is the well-known Hirota bilinear operator

$$D_x^l D_t^n a\cdot b = (\partial_x - \partial_{x'})^l (\partial_t - \partial_{t'})^n a(x,t) b(x',t')|_{x'=x,t'=t}.$$

This bilinear equation further suggests

$$f_{xt}^{(1)} + h_1 f_{xxxx}^{(1)} + 4h_2 f_{xx}^{(1)} - xh_3 f_{xx}^{(1)} - h_3 f_x^{(1)} = 0, \tag{1.2.3a}$$

$$2f_{xt}^{(2)} + 2h_1 f_{xxxx}^{(1)} + 8h_2 f_{xx}^{(2)} - 2xh_3 f_{xx}^{(2)} - 2h_3 f_x^{(2)}$$
$$= -D_x D_t f^{(1)}\cdot f^{(1)} - h_1 D_x^4 f^{(1)}\cdot f^{(1)} - 4h_2 D_x^2 f^{(1)}\cdot f^{(1)} + xh_3 D_x^2 f^{(1)}\cdot f^{(1)} + 2h_3 f^{(1)}\cdot f_x^{(1)}, \tag{1.2.3b}$$

$$\cdots\cdots$$

under the perturbation expansion

$$f(x,t) = 1 + f^{(1)}\epsilon + f^{(2)}\epsilon^2 + f^{(3)}\epsilon^3 + \cdots. \tag{1.2.4}$$

Taking

$$f^{(1)} = \omega_1(t) e^{\xi_1},\quad \xi_1 = k_1(t)x + \xi_1^{(0)}, \tag{1.2.5a}$$

from eq. (1.2.3), we obtain

$$k_{1,t}(t) = h_3 k_1(t),\quad \omega_{1,t}(t) = -h_1\omega_1(t)k_1^3(t) - 4h_2\omega_1(t)k_1(t), \tag{1.2.5b}$$

and

$$f^{(j)} = 0,\quad j = 2,3,\cdots. \tag{1.2.5c}$$

Thus, the one-soliton solution for the vcKdV equation is

$$u = \frac{k_1^2(t)}{2}\mathrm{sech}^2\frac{\xi_1}{2}. \tag{1.2.6}$$

Similar to the one-soliton solution, if we take

$$f^{(1)} = \omega_1(t)e^{\xi_1} + \omega_2(t)e^{\xi_2}, \ \ \xi_j = k_j(t)x + \xi_j^{(0)}, \ \ (j = 1,2) \tag{1.2.7a}$$

then

$$f^{(2)} = \omega_1(t)\omega_2(t)e^{\xi_1+\xi_2+A_{12}}, \tag{1.2.7b}$$

$$e^{A_{12}} = \left(\frac{k_1(t) - k_2(t)}{k_1(t) + k_2(t)}\right)^2, \ \ k_{j,t}(t) = h_3k_j(t), \ \ \omega_{j,t}(t) = -h_1\omega_j(t)k_j^3(t) - 4h_2\omega_j(t)k_j(t), \ \ (j = 1,2), \tag{1.2.7c}$$

and

$$f^{(j)} = 0, \ \ j = 3,4,\cdots. \tag{1.2.7d}$$

Therefore, the two-soliton solution is obtained from eq. (1.1.1), where

$$f = 1 + \omega_1(t)e^{\xi_1} + \omega_2(t)e^{\xi_2} + \omega_1(t)\omega_2(t)e^{\xi_1+\xi_2+A_{12}}. \tag{1.2.8}$$

This process can be extended to the three-soliton solution, four-soliton solution and so on. Generally, we obtain

$$f = \sum_{\mu=0,1} \exp[\sum_{j=1}^{N} \mu_j(\xi_j + \ln\omega_j(t)) + \sum_{1\leq j<l}^{N} \mu_j\mu_l A_{jl}], \tag{1.2.9a}$$

$$\xi_j = k_j(t)x + \xi_j^{(0)}, \ \ e^{A_{jl}} = \left(\frac{k_j(t) - k_l(t)}{k_j(t) + k_l(t)}\right)^2, \tag{1.2.9b}$$

$$k_{j,t}(t) = h_3k_j(t), \ \ \omega_{j,t}(t) = -h_1\omega_j(t)k_j^3(t) - 4h_2\omega_j(t)k_j(t), \tag{1.2.9c}$$

where the sum is obtain over all possible combinations of $\mu_j = 0,1$ $(j = 1,2,\cdots,N)$.

1.3 Exact solutions in the Wronskian form

In this section, we show that the bilinear equation (1.2.2) has a solution in the following Wronskian form

$$f = W(\phi_1,\phi_2,\cdots,\phi_N) = \begin{vmatrix} \phi_1 & \partial\phi_1 & \cdots & \partial^{N-1}\phi_1 \\ \phi_2 & \partial\phi_2 & \cdots & \partial^{N-1}\phi_2 \\ \cdots & \cdots & \cdots & \cdots \\ \phi_N & \partial\phi_N & \cdots & \partial^{N-1}\phi_N \end{vmatrix} = |0,1,\cdots,N-1| = |\widehat{N-1}|, \tag{1.3.1}$$

where the entries $\phi_j (j = 1, 2, \cdots, N)$ are under the following conditions

$$\phi_{j,xx} = \frac{k_j^2(t)}{4}\phi_j, \tag{1.3.2a}$$

$$\phi_{j,t} = -4h_1\phi_{j,xxx} - 4h_2\phi_{j,x} + xh_3\phi_{j,x}, \tag{1.3.2b}$$

We observe that

$$f_x = |\widehat{N-2}, N|, \quad f_{xx} = |\widehat{N-3}, N-1, N| + |\widehat{N-2}, N+1|, \tag{1.3.3a}$$

$$f_{xxx} = |\widehat{N-4}, N-2, N-1, N| + 2|\widehat{N-3}, N-1, N+1| + |\widehat{N-2}, N+2|, \tag{1.3.3b}$$

$$f_{xxxx} = |\widehat{N-5}, N-3, N-2, N-1, N| + 3|\widehat{N-4}, N-2, N-1, N+1|$$
$$+2|\widehat{N-3}, N, N+1| + 3|\widehat{N-3}, N-1, N+2| + |\widehat{N-2}, N+3|. \tag{1.3.3c}$$

From (1.3.2b), we have

$$f_t = -4h_1[|\widehat{N-4}, N-2, N-1, N| - |\widehat{N-3}, N-1, N+1| + |\widehat{N-2}, N+2|]$$
$$-4h_2 f_x + xh_3 f_x + \frac{N(N-1)}{2}h_3 f, \tag{1.3.4a}$$

$$f_{tx} = -4h_1[|\widehat{N-5}, N-3, N-2, N-1, N| - |\widehat{N-3}, N, N+1| + |\widehat{N-2}, N+3|]$$
$$-4h_2 f_{xx} + xh_3 f_{xx} + h_3 f_x + \frac{N(N-1)}{2}h_3 f_x, \tag{1.3.4b}$$

Substitution of (1.3.1) in (1.2.2) gives

$$f_{xt}f - f_x f_t + h_1(f_{xxxx}f - 4f_{xxx}f_x + 3f_{xx}^2) + 4h_2(f_{xx}f - f_x^2) - xh_3(f_{xx}f - f_x^2) - h_3 f f_x$$
$$= 12h_1(|\widehat{N-3}, N-2, N-1||\widehat{N-3}, N, N+1| + |\widehat{N-3}, N-2, N||\widehat{N-3}, N-1, N+1|$$
$$-|\widehat{N-3}, N-1, N||\widehat{N-3}, N-2, N+1|) = 0, \tag{1.3.5}$$

where we have make the use of the fact

$$\{\sum_{j=1}^{N}\frac{k_j^2(t)}{4}[\sum_{j=1}^{N}\frac{k_j^2(t)}{4}|\widehat{N-1}|]\}|\widehat{N-1}| = [\sum_{j=1}^{N}\frac{k_j^2(t)}{4}|\widehat{N-1}|]^2, \tag{1.3.6a}$$

or

$$|\widehat{N-1}|[|\widehat{N-5}, N-3, N-2, N-1, N| - |\widehat{N-4}, N-2, N-1, N+1| + 2|\widehat{N-3}, N, N+1|$$
$$-|\widehat{N-3}, N-1, N+2| + |\widehat{N-2}, N+3|] = [-|\widehat{N-3}, N-1, N| + |\widehat{N-2}, N+1|]^2. \tag{1.3.6b}$$

Thus we have complete the verifications.

An explicit form of ϕ_j which meets the conditions (1.3.2a) and (1.3.2b) can be given as

$$\phi_j = a_j(t)e^{\frac{\xi_j}{2}} + (-1)^{j-1}b_j(t)e^{-\frac{\xi_j}{2}}, \quad j = 1, 2, \cdots, N, \tag{1.3.7a}$$

$$a_{j,t}(t) = -\frac{1}{2}h_1k_j^3(t)a_j(t) - 2h_2k_j(t)a_j(t), \quad b_{j,t}(t) = \frac{1}{2}h_1k_j^3(t)b_j(t) + 2h_2k_j(t)b_j(t), \tag{1.3.7b}$$

Then similar to Ref.[14], the Wronskian (1.3.1) can be written as

$$f = (\frac{1}{2})^{\frac{N(N-1)}{2}} \sum_{\epsilon=\pm1} \prod_{1\leq j<l}^{N} \epsilon_l(\epsilon_j k_j(t) - \epsilon_l k_l(t)) \exp[\frac{1}{2}\sum_{j=1}^{N}(\epsilon_j\xi_j + \frac{(\epsilon_j+1)}{2}\ln a_j(t) + \frac{(1-\epsilon_j)}{2}\ln b_j(t))]. \tag{1.3.8}$$

1.4 BT for the vcKdV equation

In this section, we first derive a bilinear BT of eq.(1.2.1) from the Lax pair

$$\phi_{xx} = (\lambda - u)\phi, \tag{1.4.1a}$$

$$\phi_t = h_1u_x\phi + [xh_3 - 2h_1(u_1 + 2\lambda) - 4h_2]\phi_x - h_3(N-1)\phi, \tag{1.4.1b}$$

and then find solutions with the help of the obtained bilinear BT.

Through the transformation (1.2.1) and $\phi = \frac{g}{f}$, it is not difficult to derive the following bilinear form

$$D_x^2 g \cdot f = \lambda gf, \tag{1.4.2a}$$

$$D_t g \cdot f + h_1(D_x^3 + 3\lambda D_x)g \cdot f - xh_3D_xg \cdot f + 4h_2D_xg \cdot f - h_3(N-1)gf = 0, \tag{1.4.2b}$$

A zero soliton solution corresponds to $f = 1$. Then, substituting $f = 1$ into (1.4.2a) and (1.4.2b)($N = 1$), we have

$$g_{xx} = \lambda g, \tag{1.4.3a}$$

$$g_t + h_1(g_{xxx} + 3\lambda g_x) - xh_3g_x + 4h_2g_x = 0. \tag{1.4.3b}$$

Let $\lambda = \frac{k_1^2(t)}{4}$, then

$$g = g_1 = a_1(t)e^{\frac{\xi_1}{2}} + b_1(t)e^{-\frac{\xi_1}{2}}, \tag{1.4.4a}$$

$$k_{1,t}(t) = h_3k_1(t), \quad a_{1,t} = -\frac{1}{2}a_1(t)h_1k_1^3(t) - 2h_2a_1(t)k_1(t), \quad b_{1,t} = \frac{1}{2}b_1(t)h_1k_1^3(t) + 2h_2b_1(t)k_1(t), \tag{1.4.4b}$$

which is the one soliton solution to the vcKdV equation.

If we take $f = g_1$, from (1.4.1)($N = 2$), we can obtain the two soliton solution for the vcKdV equation (1.1.1) where

$$g = g_2 = c_1(t)[a_1(t)a_2(t)e^{\frac{\xi_1+\xi_2}{2}} + b_1(t)b_2(t)e^{\frac{-\xi_1-\xi_2}{2}}] + c_2(t)[a_1(t)b_2(t)e^{\frac{\xi_1-\xi_2}{2}} + b_1(t)a_2(t)e^{\frac{-\xi_1+\xi_2}{2}}], \tag{1.4.5a}$$

$$c_1(t) = k_1(t) - k_2(t), \quad c_2(t) = -(k_1(t) + k_2(t)), \quad k_{j,t}(t) = h_3 k_j(t),$$

$$a_{j,t}(t) = -\frac{1}{2}a_j(t)h_1k_j^3(t) - 2h_2a_j(t)k_j(t), \quad b_{j,t} = \frac{1}{2}b_j(t)h_1k_j^3(t) + 2h_2b_j(t)k_j(t), \quad j = 1, 2. \tag{1.4.5b}$$

Similar to the two soliton solution. Taking $f = g_2$, from (1.4.1)($N = 3$), we can obtain

$$\begin{aligned} g = g_3 = {} & c_3(t)[a_1(t)a_2(t)a_3(t)e^{\frac{\xi_1+\xi_2+\xi_3}{2}} + b_1(t)b_2(t)b_3(t)e^{\frac{-\xi_1-\xi_2-\xi_3}{2}}] \\ & + c_4(t)[a_1(t)a_2(t)b_3(t)e^{\frac{\xi_1+\xi_2-\xi_3}{2}} + b_1(t)b_2(t)a_3(t)e^{\frac{-\xi_1-\xi_2+\xi_3}{2}}] \\ & + c_5(t)[a_1(t)b_2(t)a_3(t)e^{\frac{\xi_1-\xi_2+\xi_3}{2}} + b_1(t)a_2(t)b_3(t)e^{\frac{-\xi_1+\xi_2-\xi_3}{2}}] \\ & + c_6(t)[b_1(t)a_2(t)a_3(t)e^{\frac{-\xi_1+\xi_2+\xi_3}{2}} + a_1(t)b_2(t)b_3(t)e^{\frac{\xi_1-\xi_2-\xi_3}{2}}], \end{aligned} \tag{1.4.6a}$$

$$c_3(t) = (k_1(t) - k_2(t))(k_1(t) - k_3(t))(k_2(t) - k_3(t)),$$

$$c_4(t) = (k_1(t) - k_2(t))(k_1(t) + k_3(t))(k_2(t) + k_3(t)), \tag{1.4.6b}$$

$$c_5(t) = (k_1(t) + k_2(t))(k_1(t) - k_3(t))(k_2(t) + k_3(t)),$$

$$c_6(t) = (k_1(t) + k_2(t))(k_1(t) + k_3(t))(k_2(t) - k_3(t)), \tag{1.4.6c}$$

$$k_{j,t}(t) = h_3k_j(t), \quad a_{j,t}(t) = -\frac{1}{2}a_j(t)h_1k_j^3(t) - 2h_2a_j(t)k_j(t),$$

$$b_{j,t} = \frac{1}{2}b_j(t)h_1k_j^3(t) + 2h_2b_j(t)k_j(t), j = 1, 2, 3. \tag{1.4.6d}$$

Generally, taking $f = g_{N-1}$, we can obtain

$$g = g_N = \sum_{\mu=\pm1}\prod_{1\leq j<l}^{N} \mu_l(\mu_jk_j(t) - \mu_lk_l(t))\exp[\frac{1}{2}\sum_{j=1}^{N}(\mu_j\xi_j + \frac{\mu_j+1}{2}\ln a_j(t) + \frac{1-\mu_j}{2}\ln b_j(t))], \tag{1.4.7a}$$

$$k_{j,t}(t) = h_3k_j(t), \quad a_{j,t}(t) = -\frac{1}{2}a_j(t)h_1k_j^3(t) - 2h_2a_j(t)k_j(t), \quad b_{j,t} = \frac{1}{2}b_j(t)h_1k_j^3(t) + 2h_2b_j(t)k_j(t). \tag{1.4.7b}$$

Finally, we obtain the Wronskian form solution for the bilinear BT (1.4.2).

Let

$$g = |\widehat{N-1}|, \quad f = |\widehat{N-2}, \tau|, \quad \tau = |\cdots, 0, 1|^T, \tag{1.4.8}$$

where ϕ_j satisfies (1.3.2) and g, f denote the N-soliton solution and $(N-1)$-soliton solution of vcKdV equation, respectively. Substitution of these functions in (1.4.1) and using

$$\frac{k_N^2(t)}{4} gf = (\sum_{j=1}^{N} \frac{k_j^2(t)}{4} g) f - g(\sum_{j=1}^{N-1} \frac{k_j^2(t)}{4} f)$$

$$= (-|\widehat{N-3}, N-1, N| + |\widehat{N-2}, N+1|)|\widehat{N-2}, \tau| + |\widehat{N-1}|(|\widehat{N-4}, N-2, N-1, \tau| - |\widehat{N-3}, N, \tau|), \tag{1.4.9a}$$

$$\frac{k_N^2(t)}{4} g_x f = (\sum_{j=1}^{N} \frac{k_j^2(t)}{4} g_x) f - g_x(\sum_{j=1}^{N-1} \frac{k_j^2(t)}{4} f)$$

$$= -(|\widehat{N-4}, N-2, N-1, N| + |\widehat{N-2}, N+2|)|\widehat{N-2}, \tau|$$

$$+|\widehat{N-2}, N|(|\widehat{N-4}, N-2, N-1, \tau| - |\widehat{N-3}, N, \tau|, \tag{1.4.9b}$$

$$\frac{k_N^2(t)}{4} g f_x = (\sum_{j=1}^{N} \frac{k_j^2(t)}{4} g) f_x - g(\sum_{j=1}^{N-1} \frac{k_j^2(t)}{4} f_x)$$

$$= (-|\widehat{N-3}, N-1, N| + |\widehat{N-2}, N+1|)(|\widehat{N-3}, N-1, \tau|$$

$$+|\widehat{N-1}|(|\widehat{N-5}, N-3, N-2, N-1, \tau| - |\widehat{N-3}, N+1, \tau|), \tag{1.4.9c}$$

we can obtain

$$|\widehat{N-3}, N-1, N||\widehat{N-2}, \tau| - |\widehat{N-2}, N||\widehat{N-3}, N-1, \tau| + |\widehat{N-3}, N, \tau||\widehat{N-1}| = 0, \tag{1.4.10a}$$

$$6h_1(t)(|\widehat{N-3}, N-1, N+1||\widehat{N-2}, \tau| + |\widehat{N-3}, N+1, \tau||\widehat{N-1}|$$

$$-|\widehat{N-2}, N+1||\widehat{N-3}, N-1, \tau| - |\widehat{N-4}, N-2, N-1, N||\widehat{N-2}, \tau|$$

$$+|\widehat{N-2}, N||\widehat{N-4}, N-2, N-1, \tau| - |\widehat{N-4}, N-2, N, \tau||\widehat{N-1}|) = 0. \tag{1.4.10b}$$

Chapter 2
The Exact Solutions for a Nonisospectral and Variable-coefficient KP Equation

The bilinear form for a nonisospectral and variable-coefficient KP equation is obtained and some exact soliton solutions are derived through Hirota method and Wronskian technique. We also derive the bilinear Bäcklund transformation from its Lax pairs and find solutions with the help of the obtained bilinear Bäcklund transformation.

2.1 Introduction

In recent years, much attention has been paid on the study of nonlinear differential equations with variable coefficients[1,2,4-6]. Chan, Zheng[4] and Chan, Li[5] studied the nonisospectral and variable-coefficient KdV equation by the method of Bäcklund transformation and inverse scattering. In 1992, Chan *et al.*[6] obtained the n-soliton solutions for a nonisospectral variable-coefficient KP (vcKP) equation by the dressing method and studied the interactions of two-soliton solution.

The Hirota method[10], Bäcklund transformation(BT)[11] and Wronskian technique[12] are three efficient direct ways to find soliton solutions for nonlinear equations. Recently, Zhang *et al.*[13] study the soliton for the KdV equation with loss and non-uniformity terms by use of Hirota method and Wronskian technique. In this paper, we would like to consider the vcKP equation through above three methods. The bilinear form of the vcKP equation is given and one-, two-soliton solutions are obtained through the standard Hirota method. A general formula which denotes higher order solutions is also given. In a way similar to the isospectral equation, from the Lax pairs we can derive the bilinear BT for the vcKP equation by the variable transformations. However it is not an auto-Bäcklund transformation. We also obtain the solution in Wronskian form. The methods used here can be applied to other nonisospectral soliton equations.

The paper is organized as follows. In Sec.2, we solve the vcKP equation by the Hirota method. In Sec.3 solution in Wronskian form is proven. In Sec.4, the soliton solutions for the vcKP equation are obtained by bilinear BT.

2.2 Bilinear form and Hirota method

We consider the generalized variable-coefficient KP equation

$$u_t = h(u_{xxx}+6uu_x+3\alpha^2 w_{yy})+b_1u_x-k(xu_x+2u+2yu_y)-\alpha b_1 xu_y-2\alpha b_1 w_y,\quad w_x=u, \tag{2.2.1}$$

and its Lax pairs

$$\alpha\phi_y = \phi_{xx}+u\phi, \tag{2.2.2a}$$

$$\phi_t + A\phi + B\phi_x + D\phi_{xx} + E\phi_{xxx} = 0, \tag{2.2.2b}$$

$$h=-yb_1(t)/2\alpha - b_2(t)/4,\quad E=-4h,\quad D=xb_1(t)+2yk(t)/\alpha, \tag{2.2.2c}$$

$$B=xk(t)-6hu,\quad A=-3hu_x+wb_1(t)/2-3\alpha hw_y+Du-(N-1). \tag{2.2.2d}$$

Through the transformation

$$u=2(\ln f)_{xx}, \tag{2.2.3}$$

eq.(2.2.1) can be transformed into the bilinear form

$$h(D_x^4 f\cdot f+3\alpha^2 D_y^2 f\cdot f)+b_1D_x^2 f\cdot f-k(xD_x^2 f\cdot f+2f_xf+2yD_xD_y f\cdot f)$$

$$-\alpha b_1 xD_xD_y f\cdot f-2\alpha b_1 f_yf-D_xD_t f\cdot f=0, \tag{2.2.4}$$

where D is the well-known Hirota bilinear operator

$$D_x^l D_y^m D_t^n a\cdot b=(\partial_x-\partial_{x'})^l(\partial_y-\partial_{y'})^m(\partial_t-\partial_{t'})^n a(x,y,t)b(x',y',t')|_{x'=x,y'=y,t'=t}.$$

We expand f into power series of a small parameter ϵ as

$$f=1+f^{(1)}\epsilon+f^{(2)}\epsilon^2+f^{(3)}\epsilon^3+\cdots. \tag{2.2.5}$$

Substituting (2.2.5) into (2.2.4) and equating coefficients of ϵ yield

$$h(f^{(1)}_{xxxx}+3\alpha^2 f^{(1)}_{yy})+b_1f^{(1)}_{xx}-k(xf^{(1)}_{xx}+f^{(1)}_x+2yf^{(1)}_{xy})-\alpha b_1 xf^{(1)}_{xy}-\alpha b_1 f^{(1)}_y-f^{(1)}_{xt}=0, \tag{2.2.6a}$$

$$2h(f^{(2)}_{xxxx}+3\alpha^2 f^{(2)}_{yy})+2b_1f^{(2)}_{xx}-2k(xf^{(2)}_{xx}+f^{(2)}_x+2yf^{(2)}_{xy})-2\alpha b_1 xf^{(2)}_{xy}-2\alpha b_1 f^{(2)}_y-2f^{(2)}_{xt}$$

$$=-h(D_x^4 f^{(1)}\cdot f^{(1)}+3\alpha^2 D_y^2 f^{(1)}\cdot f^{(1)})-b_1D_x^2 f^{(1)}\cdot f^{(1)}+k(xD_x^2 f^{(1)}\cdot f^{(1)}$$

$$+2f^{(1)}_x f^{(1)}+2yD_xD_y f^{(1)}\cdot f^{(1)})+\alpha b_1 xD_xD_y f^{(1)}\cdot f^{(1)}+2\alpha b_1 f^{(1)}_y f^{(1)}+D_xD_t f^{(1)}\cdot f^{(1)}, \tag{2.2.6b}$$

$$\cdots\cdots$$

Taking

$$f^{(1)} = w_1(t)e^{\xi_1+\eta_1}, \quad \xi_1 = p_1(t)x - p_1^2(t)y/\alpha + \xi_1^{(0)}, \quad \eta_1 = q_1(t)x + q_1^2(t)y/\alpha + \eta_1^{(0)}, \qquad (2.2.7a)$$

from (2.2.6), we have

$$p_{1,t}(t) = -kp_1(t) + b_1 p_1^2(t), \quad q_{1,t}(t) = -kq_1(t) - b_1 q_1^2(t),$$

$$w_{1,t}(t) = b_1 w_1(t)[p_1(t) + q_1(t)] - b_2 w_1(t)[p_1^3(t) + q_1^3(t)], \qquad (2.2.7b)$$

and

$$f^{(j)} = 0, \quad j = 2, 3, \cdots. \qquad (2.2.7c)$$

So the one-soliton solution for the vcKP equation is

$$u = \frac{[p_1(t) + q_1(t)]^2}{2} \mathrm{sech}^2 \frac{\xi_1 + \eta_1 + \ln w_1(t)}{2}. \qquad (2.2.8)$$

From eq.(2.2.7-8), we know that at time t, the one soliton achieve its peak value

$$V_p(t) = \frac{[p_1(t) + q_1(t)]^2}{2}, \qquad (2.2.9)$$

on the line, in xy plane, defined by the equation

$$[q_1^2(t) - p_1^2(t)]/\alpha y + [p_1(t) + q_1(t)x] + \ln w_1(t) = 0. \qquad (2.2.10)$$

The velocity of the one soliton at time t has the components

$$v_x(t) = (\frac{\partial}{\partial t})\{-(\ln w_1(t))/[p_1(t) + q_1(t)] - [q_1(t) - p_1(t)]y/\alpha\}$$

and

$$v_y(t) = -\alpha(\frac{\partial}{\partial t})\{x/[q_1(t) - p_1(t)] + (\ln w_1(t))/[q_1^2(t) - p_1^2(t)]\}.$$

Similar to the one-soliton solution, if we take

$$f^{(1)} = w_1(t)e^{\xi_1+\eta_1} + w_2(t)e^{\xi_2+\eta_2},$$

$$\xi_j = p_j(t)x - p_j^2(t)y/\alpha + \xi_j^{(0)}, \quad \eta_j = q_j(t)x + q_j^2(t)y/\alpha + \eta_j^{(0)}, \quad j = 1, 2, \qquad (2.2.11a)$$

then

$$f^{(2)} = w_1(t)w_2(t)e^{\xi_1+\xi_2+\eta_1+\eta_2+A_{12}}, \quad e^{A_{12}} = \frac{[p_1(t) - p_2(t)][q_1(t) - q_2(t)]}{[p_1(t) + q_2(t)][p_2(t) + q_1(t)]}, \qquad (2.2.11b)$$

$$p_{j,t}(t) = -kp_j(t) + b_1 p_j^2(t), \quad q_{j,t}(t) = -kq_j(t) - b_1 q_j^2(t),$$

$$w_{j,t}(t) = b_1 w_j(t)[p_j(t) + q_j(t)] - b_2 w_j(t)[p_j^3(t) + q_j^3(t)], \quad j = 1, 2, \tag{2.2.11c}$$

$$f^{(j)} = 0, \quad j = 3, 4, \cdots. \tag{2.2.11d}$$

Therefore the two-soliton solution is obtained from (2.2.3) here

$$f = 1 + w_1(t)e^{\xi_1+\eta_1} + w_2(t)e^{\xi_2+\eta_2} + w_1(t)w_2(t)e^{\xi_1+\eta_1+\xi_2+\eta_2+A_{12}}. \tag{2.2.12}$$

Decomposition of the two soliton solutions and their interactions are studied in detail in Ref[6].

This process can be continued to the three-, four-soliton solutions and so on. Generally, we have

$$f = \sum_{\epsilon=0,1} \exp[\sum_{j=1}^{N} \epsilon_j(\xi_j + \eta_j + \ln w_j(t)) + \sum_{1\le j<l}^{N} \epsilon_j \epsilon_l A_{jl}], \tag{2.2.13a}$$

$$\xi_j = p_j(t)x - p_j^2(t)y/\alpha + \xi_j^{(0)}, \quad \eta_j = q_j(t)x + q_j^2(t)y/\alpha + \eta_j^{(0)}, \quad e^{A_{jl}} = \frac{[p_l(t) - p_j(t)][q_l(t) - q_j(t)]}{[p_l(t) + q_j(t)][p_j(t) + q_l(t)]}. \tag{2.2.13b}$$

$$p_{j,t}(t) = -kp_j(t) + b_1 p_j^2(t), \quad q_{j,t}(t) = -kq_j(t) - b_1 q_j^2(t),$$

$$w_{j,t}(t) = b_1 w_j(t)[p_j(t) + q_j(t)] - b_2 w_j(t)[p_j^3(t) + q_j^3(t)]. \tag{2.2.13c}$$

where the sum is taken over all possible combinations of $\epsilon_j = 0, 1$ $(j = 1, 2, \cdots, N)$.

2.3 Exact solutions in the Wronskian form

In the present section, we show that the bilinear equation (2.4) has a solution in the following Wronskian form

$$f = W(\phi_1, \phi_2, \cdots, \phi_N) = \begin{vmatrix} \phi_1 & \partial\phi_1 & \cdots & \partial^{N-1}\phi_1 \\ \phi_2 & \partial\phi_2 & \cdots & \partial^{N-1}\phi_2 \\ \cdots & \cdots & \cdots & \cdots \\ \phi_N & \partial\phi_N & \cdots & \partial^{N-1}\phi_N \end{vmatrix} = |0, 1, \cdots, N-1| = |\widehat{N-1}|, \tag{2.3.1}$$

where the entries $\phi_j (j = 1, 2, \cdots, N)$ enjoy the following conditions

$$\phi_{j,y} = -\frac{1}{\alpha}\phi_{j,xx}, \tag{2.3.2a}$$

$$\phi_{j,t} = 4h\phi_{j,xxx} + \frac{2k}{\alpha}y\phi_{j,xx} + xb_1\phi_{j,xx} - xk\phi_{j,x} - (N-2)b_1\phi_{j,x}. \tag{2.3.2b}$$

We observe that

$$f_x = |\widehat{N-2}, N|, \quad f_{xx} = |\widehat{N-3}, N-1, N| + |\widehat{N-2}, N+1|, \tag{2.3.3a}$$

$$f_{xxx}=|\widehat{N-4},N-2,N-1,N|+2|\widehat{N-3},N-1,N+1|+|\widehat{N-2},N+2|, \tag{2.3.3b}$$

$$f_{xxxx}=|\widehat{N-5},N-3,N-2,N-1,N|+3|\widehat{N-4},N-2,N-1,N+1|$$

$$+2|\widehat{N-3},N,N+1|+3|\widehat{N-3},N-1,N+2|+|\widehat{N-2},N+3|. \tag{2.3.3c}$$

Using $(2.3.2a)$ we have the y derivative

$$f_y=\frac{1}{\alpha}(|\widehat{N-3},N-1,N|-|\widehat{N-2},N+1|), \tag{2.3.4a}$$

$$f_{yy}=\frac{1}{\alpha^2}(|\widehat{N-5},N-3,N-2,N-1,N|-|\widehat{N-3},N-1,N+2|$$

$$+2|\widehat{N-3},N,N+1|-|\widehat{N-4},N-2,N-1,N+1|+|\widehat{N-2},N+3|). \tag{2.3.4b}$$

From $(2.3.2b)$ and (2.3.4), we have

$$f_t=4h(|\widehat{N-4},N-2,N-1,N|-|\widehat{N-3},N-1,N+1|+|\widehat{N-2},N+2|)$$

$$-2yk/\alpha(|\widehat{N-3},N-1,N|-|\widehat{N-2},N+1|)-xk|\widehat{N-2},N|-\frac{N(N-1)}{2}k|\widehat{N-1}|$$

$$-xb_1(|\widehat{N-3},N-1,N|-|\widehat{N-2},N+1|)+b_1|\widehat{N-2},N|, \tag{2.3.5a}$$

$$f_{tx}=4h(|\widehat{N-5},N-3,N-2,N-1,N|-|\widehat{N-3},N,N+1|+|\widehat{N-2},N+3|)$$

$$-2yk/\alpha(|\widehat{N-4},N-2,N-1,N|-|\widehat{N-2},N+2|)-xk(|\widehat{N-3},N-1,N|+|\widehat{N-2},N+1|)$$

$$-k|\widehat{N-2},N|-\frac{N(N-1)}{2}k|\widehat{N-2},N|-b_1(|\widehat{N-3},N-1,N|-|\widehat{N-2},N+1|)$$

$$-xb_1(|\widehat{N-4},N-2,N-1,N|-|\widehat{N-2},N+2|)+b_1(|\widehat{N-3},N-1,N|+|\widehat{N-2},N+1|), \tag{2.3.5b}$$

Substitution of (2.3.1) in (2.2.4) gives

$$2h(f_{xxxx}f-4f_xf_{xxx}+3f_{xx}^2+3\alpha^2f_{yy}f-3\alpha^2f_y^2)+2b_1(f_{xx}f-f_x^2)$$

$$-2xk(f_{xx}f-f_x^2)-2kf_xf-4yk(f_{xy}f-f_xf_y)-2x\alpha b_1f_yf-2(f_{xt}f-f_xf_t)$$

$$=2h(f_{xxxx}f-4f_xf_{xxx}+3f_{xx}^2+3\alpha^2f_{yy}f-3\alpha^2f_y^2)$$

$$-8h(|\widehat{N-5},N-3,N-2,N-1,N|-|\widehat{N-3},N,N+1|+|\widehat{N-2},N+3|)|\widehat{N-1}|$$

$$+8h(|\widehat{N-4},N-2,N-1,N|-|\widehat{N-3},N-1,N+1|+|\widehat{N-2},N+2|)|\widehat{N-2},N|$$

$$=24h(|\widehat{N-3},N-2,N-1||\widehat{N-3},N,N+1|-|\widehat{N-3},N-2,N||\widehat{N-3},N-1,N+1|$$

$$+|\widehat{N-3},N-2,N+1||\widehat{N-3},N-1,N|)=0. \tag{2.3.6}$$

So the eq.(2.3.1) with (2.3.2) solve (2.2.4).

Let us define the entries

$$\phi_j = \alpha_j^+ A_j(t) e^{\xi_j} + \alpha_j^- B_j(t) e^{-\eta_j}, \quad (j = 1, 2, \cdots, N) \tag{2.3.7a}$$

$$A_{j,t}(t) = -b_2 p_j^3(t) A_j(t) - (N-2) b_1 p_j(t) A_j(t), \quad B_{j,t}(t) = b_2 q_j^3(t) B_j(t) + (N-2) b_1 q_j(t) B_j(t), \tag{2.3.7b}$$

If we take $\alpha_j^+ = 1$ and $\alpha_j^- = (-1)^{j-1}$, then similar to Ref.[14], the Wronskian (2.3.1) can be written as

$$f = \prod_{1\le j<l}^{N} [q_l(t) - q_j(t)] \exp[\sum_{j=1}^{N} (-\eta_j + \ln B_j(t))] \sum_{\epsilon=0,1} \exp[\sum_{j=1}^{N} \epsilon_j (\xi_j' + \eta_j') + \sum_{1\le j<l}^{N} \epsilon_j \epsilon_l A_{jl}], \tag{2.3.8a}$$

$$\xi_j' = \xi_j + \ln A_j(t) + \sum_{j\neq l} [p_j(t) + q_l(t)], \quad \eta_j' = \eta_j - \ln B_j(t) + \sum_{j>l} [q_j(t) - q_l(t)]^{-1} + \sum_{l>j} [q_l(t) - q_j(t)]^{-1}, \tag{2.3.8b}$$

So we give another form of solution (2.3.1). However, the solutions (2.3.8) and (2.2.13) are slightly different for the covering of the N-soliton solution from the transformation (2.2.3). One can find that there is time-dependent initial phase in each $\xi_j' + \eta_j'$ in the solution (2.3.8), which is different from (2.2.13).

2.4 BT for the vcKP equation

In this section, we first derive a bilinear BT of eq.(2.2.1) from the Lax pair (2.2.2a) and (2.2.2b), and then find solutions with the help of the obtained bilinear BT.

Through the transformation (2.2.1) and $\phi = \frac{g}{f}$, it is not difficult to derive the following bilinear form

$$\alpha D_y g \cdot f = D_x^2 g \cdot f, \tag{2.4.1a}$$

$$D_t g\cdot f - h(D_x^3 + 3\alpha D_x D_y) g\cdot f + (xb_1 + 2yk/\alpha) D_x^2 g\cdot f + b_1 f_x g + xk D_x g\cdot f - (N-1) kgf = 0, \tag{2.4.1b}$$

which is just the BT for the vcKP equation (2.2.1).

A zero soliton solution corresponds to $g = 1$. Then, substituting $g = 1$ into (2.4.1a) and (2.4.1b)($N = 1$), we have

$$-\alpha f_t = f_{xx}, \tag{2.4.2a}$$

$$-f_t - h(-f_{xxx} + 3\alpha f_{xy}) + (xb_1 + 2yk/\alpha) f_{xx} + b_1 f_x - xk f_x = 0. \tag{2.4.2b}$$

Its solution can be given by

$$f = f_1 = A_1(t)e^{\xi_1} + B_1(t)e^{-\eta_1}, \tag{2.4.3a}$$

where ξ_1, η_1 are defined by (2.2.11b) and

$$A_{1,t}(t) = b_1 p_1(t)A_1(t) - b_2 p_1^3(t)A_1(t), \quad B_{1,t}(t) = -b_1 q_1(t)B_1(t) + b_2 q_1^3(t)B_1(t). \tag{2.4.3b}$$

So one-soliton solution for the vcKP equation (2.2.1) is

$$u = 2(\ln f_1)_{xx} = 2[\ln(A_1(t)e^{\xi_1} + B_1(t)e^{-\eta_1})]_{xx}$$
$$= \frac{[p_1(t)+q_1(t)]^2}{2}\mathrm{sech}^2\frac{\xi_1 + \eta_1 + \ln A_1(t) - \ln B_1(t)}{2}. \tag{2.4.4}$$

If we take $g = f_1$, from (2.4.1), we can not obtain the two soliton solution for the vcKP equation (2.2.1). Taking

$$g = A_1(t)e^{\xi_1} + B_1(t)e^{-\eta_1}, \quad A_{1,t}(t) = -b_2 p_1^3(t)A_1(t), \quad B_{1,t}(t) = b_2 q_1^3(t)B_1^{(t)}, \tag{2.4.5}$$

which is the solution to the equation

$$u_t = h(u_{xxx} + 6uu_x + 3\alpha^2 w_{yy}) - k(xu_x + 2u + 2yu_y) - \alpha b_1 x u_y - 2\alpha b_1 w_y, \quad w_x = u, \quad u = 2(\ln g)_{xx}, \tag{2.4.6}$$

then the solution of (2.2.1) generated by the bilinear BT (2.4.1)($N = 2$) is given by

$$f = f_2 = h_1(t)A_1(t)A_2(t)e^{\xi_1+\xi_2} + h_2(t)B_1(t)B_2(t)e^{-\eta_1-\eta_2}$$
$$+ h_3(t)A_1(t)B_2(t)e^{\xi_1-\eta_2} + h_4(t)A_2(t)B_1(t)e^{\xi_2-\eta_1}, \tag{2.4.7a}$$

$$h_1(t) = p_1(t) - p_2(t), \quad h_2(t) = q_1(t) - q_2(t), \quad h_3(t) = -[p_1(t) + q_2(t)], \quad h_4(t) = -[p_2(t) + q_1(t)],$$
$$A_{j,t}(t) = -b_2 p_j^3(t)A_j(t), \quad B_{j,t}(t) = b_2 q_j^3(t)B_j(t), \quad (j = 1, 2), \tag{2.4.7b}$$

and $\xi_j, \eta_j, p_j(t), q_j(t)$ are (2.2.11b, c).

Similar to the two soliton solution. If we take the solution of (2.4.6)

$$g = h_1(t)A_1(t)A_2(t)e^{\xi_1+\xi_2} + h_2(t)B_1(t)B_2(t)e^{-\eta_1-\eta_2}$$
$$+ h_3(t)A_1(t)B_2(t)e^{\xi_1-\eta_2} + h_4(t)A_2(t)B_1(t)e^{\xi_2-\eta_1}, \tag{2.4.8a}$$

$$A_{j,t}(t) = -b_2 p_j^3(t)A_j(t) - b_1 p_j(t)A_j(t), \quad B_{j,t}(t) = b_2 q_j^3(t)B_j(t) + b_1 q_j(t)B_j(t), (j = 1, 2), \tag{2.4.8b}$$

from (2.4.1)($N=3$), we can derive

$$f=f_3=h_5(t)A_1(t)A_2(t)A_3(t)e^{\xi_1+\xi_2+\xi_3}+h_6(t)A_1(t)A_2(t)B_3(t)e^{\xi_1+\xi_2-\eta_3}$$

$$+h_7(t)A_1(t)B_2(t)A_3(t)e^{\xi_1-\eta_2+\xi_3}+h_8(t)B_1(t)A_2(t)A_3(t)e^{-\eta_1+\xi_2+\xi_3}$$

$$+h_9(t)B_1(t)B_2(t)A_3(t)e^{-\eta_1-\eta_2+\xi_3}+h_{10}(t)B_1(t)A_2(t)B_3(t)e^{-\eta_1+\xi_2-\eta_3}$$

$$+h_{11}(t)A_1(t)B_2(t)B_3(t)e^{\xi_1-\eta_2-\eta_3}+h_{12}(t)B_1(t)B_2(t)B_3(t)e^{-\eta_1-\eta_2-\eta_3}\ , \qquad (2.4.9a)$$

$$h_5(t)=[p_1(t)-p_2(t)][p_2(t)-p_3(t)][p_1(t)-p_3(t)],$$

$$h_6(t)=[p_1(t)-p_2(t)][p_2(t)+q_3(t)][p_1(t)+q_3(t)],$$

$$h_7(t)=[p_1(t)+q_2(t)][p_3(t)+q_2(t)][p_1(t)-k_3(t)],$$

$$h_8(t)=[p_2(t)-p_3(t)][p_2(t)+q_1(t)][p_3(t)+q_1(t)],$$

$$h_9(t)=[p_3(t)+q_1(t)][p_3(t)+q_2(t)][q_1(t)-q_2(t)],$$

$$h_{10}(t)=[p_2(t)+q_1(t)][p_2(t)+q_3(t)][q_1(t)-q_3(t)],$$

$$h_{11}(t)=[p_1(t)+q_2(t)][q_2(t)-q_3(t)][p_1(t)+q_3(t)],$$

$$h_{12}(t)=[q_1(t)-q_2(t)][q_1(t)-q_3(t)][q_2(t)-q_3(t)], \qquad (2.4.9b)$$

$$A_{j,t}(t)=-b_2p_j^3(t)A_j(t)-b_1p_j(t)A_j(t),\quad B_{j,t}(t)=b_2q_j^3(t)B_j(t)+b_1q_j(t)B_j(t),(j=1,2,3). \qquad (2.4.9c)$$

Generally, taking the $N-1$ soliton solution of the eq.(2.4.6)

$$g=\sum_{\epsilon=0,1}\prod_{1\le j<l}^{N-1}(2\epsilon_i-1)[\epsilon_jp_j(t)+(\epsilon_j-1)q_j(t)-\epsilon_lp_l(t)-(\epsilon_l-1)q_l(t)]$$

$$\exp\{\sum_{j=1}^{N-1}[\epsilon_j(\xi_j+\ln A_j(t))+(\epsilon_j-1)(\eta_j-\ln B_j(t))]\}, \qquad (2.4.10a)$$

where

$$\xi_j=p_j(t)x-p_j^2(t)y/\alpha+\xi_j^{(0)},\quad \eta_j=q_j(t)x+q_j^2(t)y/\alpha+\eta_j^{(0)},$$

$$p_{j,t}(t)=-kp_j(t)+b_1p_j^2(t),\quad q_{j,t}(t)=-kq_j(t)-b_1q_j^2(t),$$

$$A_{j,t}(t)=-b_2p_j^3(t)A_j(t)-(N-2)b_1p_j(t)A_j(t),$$

$$B_{j,t}(t)=b_2q_j^3(t)B_j(t)+(N-2)b_1p_j(t)B_j(t),(j=1,2,\cdots,N-1), \qquad (2.4.10b)$$

we can get the N-soliton solution expressed by eq.(2.2.3) with

$$f=\sum_{\epsilon=0,1}\prod_{1\le j<l}^{N}(2\epsilon_l-1)[\epsilon_j p_j(t)+(\epsilon_j-1)q_j(t)-\epsilon_l p_l(t)-(\epsilon_l-1)q_l(t)]$$

$$\exp\{\sum_{j=1}^{N}[\epsilon_j(\xi_j+\ln A_j(t))+(\epsilon_j-1)(\eta_j-\ln B_j(t))]\}, \qquad (2.4.11a)$$

$$p_{j,t}(t)=-kp_j(t)+b_1p_j^2(t),\quad q_{j,t}(t)=-kq_j(t)-b_1q_j^2(t),$$

$$A_{j,t}(t)=-b_2p_j^3(t)A_j(t)-(N-2)b_1p_j(t)A_j(t),$$

$$B_{j,t}(t)=b_2q_j^3(t)B_j(t)+(N-2)b_1p_j(t)B_j(t),(j=1,2,\cdots,N). \qquad (2.4.11b)$$

Finally, we obtain the Wronskian form solution for the bilinear BT (2.4.1).

Let

$$f=|\widehat{N-1}|,\quad g=|\widehat{N-2},\tau|,\quad \tau=|\cdots,0,1|^T, \qquad (2.4.12)$$

where ϕ_j satisfies (2.3.2) and f,g denote the N-soliton solution of vcKP equation and $(N-1)$-soliton solution for the equation (2.4.6), respectively. Then

$$g_x=|\widehat{N-3},N-1,\tau|,\quad g_{xx}=|\widehat{N-4},N-2,N-1,\tau|+|\widehat{N-3},N,\tau|, \qquad (2.4.13a)$$

$$g_{xxx}=|\widehat{N-5},N-3,N-2,N-1,\tau|+2|\widehat{N-4}.N-2,N,\tau|+|\widehat{N-3},N+1,\tau|, \qquad (2.4.13b)$$

$$g_y=\frac{1}{\alpha}(|\widehat{N-4},N-2,N-1,\tau|-|\widehat{N-3},N,\tau|), \qquad (2.4.13c)$$

$$g_{yx}=\frac{1}{\alpha}(|\widehat{N-5},N-3,N-2,N-1,\tau|-|\widehat{N-3},N+1,\tau|), \qquad (2.4.13d)$$

$$g_t=4h(|\widehat{N-5},N-3,N-2,N-1,\tau|-|\widehat{N-4},N-2,N,\tau|+|\widehat{N-3},N+1,\tau|)$$
$$-\frac{2yk}{\alpha}(|\widehat{N-4},N-2,N-1,\tau|-|\widehat{N-3},N,\tau|)-xk|\widehat{N-3},N-1,\tau|$$
$$-\frac{(N-1)(N-2)}{2}k|\widehat{N-2},\tau|-xb_1(|\widehat{N-4},N-2,N-1,\tau|-|\widehat{N-3},N,\tau|), \qquad (2.4.13e)$$

Substitution (2.3.3-5) and (2.4.13) in (2.4.1) gives

$$2|\widehat{N-2},N||\widehat{N-3},N-1,\tau|-2|\widehat{N-1}||\widehat{N-3},N,\tau|-2|\widehat{N-2},\tau||\widehat{N-3},N-1,N|=0, \qquad (2.4.14)$$

$$6h(|\widehat{N-2},N||\widehat{N-4},N-2,N-1,\tau|-|\widehat{N-1}||\widehat{N-4},N-2,N,\tau|$$
$$-|\widehat{N-2},\tau||\widehat{N-4},N-2,N-1,N|)+6h(|\widehat{N-1}||\widehat{N-3},N+1,\tau|$$
$$+|\widehat{N-2},\tau||\widehat{N-3},N-1,N+1|-|\widehat{N-3},N-1,\tau||\widehat{N-2},N+1|)=0, \qquad (2.4.15)$$

Thus verifying that the Bäcklund equation (2.4.1) are indeed satisfied.

Chapter 3
Darboux and Bäcklund Transformations for the Nonisospectral KP Equation

Darboux transformation and Bäcklund transformation in bilinear form for the nonisospectral KP equation are first investigated. Corresponding solutions are derived by using the Bäcklund transformation in bilinear form. It has been shown that these transformations are auto-Bäcklund transformations for isospectral problems while not for nonisospectral ones.

3.1 Introduction.

It is a powerful practice to utilize the idea of Bäcklund transformation (BT)[11] and Darboux transformation (DT)[15] in constructing solutions for nonlinear evolution equations. Recently in the past decade, a unified explicit form of Bäcklund transformation can be obtained for some isospectral equations, such as KdV, mKdV and KP equations[11,15-17]. These integrable equations with constant coefficients is regarded to be highly idealized in the physical situation. However, equations with variable coefficients and nonisospectral eigenparameters may provide more realistic models, in the propagation of (small-amplitude) surface waves in straits or large channels of (slowly) varying depth and width and nonvanishing vorticity [1]. Therefore, recently there has been much interest in studying the nonisospectral and variable coefficients generalizations of completely integrable nonlinear evolution equations [1-6,9,18,19].

In this paper, N-times DT for the nonisospectral KP equation is first constructed. And then from the Lax pair of nonisospectral KP equation, a BT in bilinear form can be derived. Moreover, some exact solutions are obtained with the help of bilinear BT. It is worthwhile to mention that DT and bilinear BT are auto-Bäcklund transformations for the isospectral KP equation, but this dose not true for the nonisospectral KP equation. As a matter of fact, they transform one nonisospectral KP equation to another.

The structure of this paper is organized as follows. In section 2, N-times repeated DT for the nonisospectral KP equation is derived. In section 3, BT in bilinear form of the nonisospectral KP equation is constructed. Moreover, corresponding soliton solutions are investigated by using the bilinear BT and Wronskian technique.

3.2 Darboux transformation for the nonisospectral KP equation

The KP equation reads

$$(4u_t + 6uu_x + u_{xxx})_x + 3u_{yy} = 0, \tag{3.2.1}$$

and Lax representation for (3.2.1) is

$$\phi_y = \phi_{xx} + u\phi, \tag{3.2.2a}$$

$$4\phi_t = A(u)\phi, \tag{3.2.2b}$$

where

$$A(u) = -4\partial^3 - 6u\partial - 3(u_x + \partial^{-1}u_y), \quad \partial = \frac{\partial}{\partial x}, \quad \partial^{-1} = \int_{-\infty}^{x}. \tag{3.2.3}$$

Assume that u be the solution of KP equation (3.2.1) and denote the fixed solution of (3.2.2) by ϕ_1. Then DT is defined by[15]

$$\phi[1] = \phi_x - \frac{\phi_{1,x}}{\phi_1}\phi, \tag{3.2.4a}$$

$$u[1] = u + 2\partial^2 \ln \phi_1. \tag{3.2.4b}$$

It is known that equations (3.2.2) are covariant with respect to the action of DT (3.2.4). namely $\phi[1], u[1]$ satisfy

$$\phi_y[1] = \phi_{xx}[1] + u[1]\phi[1], \tag{3.2.5a}$$

$$4\phi_t[1] = A(u[1])\phi[1], \tag{3.2.5b}$$

and $u[1]$ satisfies KP equation (3.2.1). Equation (3.2.2b), (3.2.4a) and (3.2.5b) imply that

$$4\phi_t[1] = 4[\phi_x - \frac{\phi_{1,x}}{\phi_1}\phi]_t = (A(u)\phi)_x - (\frac{A(u)\phi_1}{\phi_1})_x\phi - \frac{\phi_{1,x}}{\phi_1}A(u)\phi = A(u[1])\phi[1]. \tag{3.2.6}$$

Thus the covariance of (3.2.2) with respect to the action of DT (3.2.4) leads to the following lemma.

Lemma 2.1: if u is the solution of KP equation (3.2.1) and ϕ_1 is a solution of (3.2.2) and DT is given by (3.2.4), then the formula (3.2.6) holds.

Based on the DT (3.2.4) for KP equation (3.2.1). Now we construct DT for nonisospectral KP equation.

Consider the nonisospectral KP equation[20]

$$4u_t + y(u_{xxx} + 6uu_x + 3\partial^{-1}u_{yy}) + 2xu_y + 4\partial^{-1}u_y = 0, \tag{3.2.7}$$

and its Lax pair

$$\phi_y = \phi_{xx} + u\phi, \tag{3.2.8a}$$

$$4\phi_t = yA(u)\phi + xB(u)\phi + C(u)\phi, \tag{3.2.8b}$$

where

$$B(u) = -2(\partial^2 + u), \quad C(u) = -2\partial - \partial^{-1}u. \tag{3.2.9}$$

Theorem 2.1: Assume that u be the solution of nonisospectral KP equation (3.2.7) and ϕ_1 satisfies (3.2.8), then the Darboux transformation is defined by

$$\phi[1] = \phi_x - \frac{\phi_{1,x}}{\phi_1}\phi, \tag{3.2.10a}$$

$$u[1] = u + 2\partial^2 \ln \phi_1, \tag{3.2.10b}$$

and DT (3.2.10) transform Lax representation (3.2.8) into the Lax representation as follows

$$\phi_y[1] = \phi_{xx}[1] + u[1]\phi[1], \tag{3.2.11a}$$

$$4\phi_t[1] = yA(u[1])\phi[1] + xB(u[1])\phi[1] + C(u[1])\phi[1] - 2\phi_x[1], \tag{3.2.11b}$$

and $u[1]$ satisfies nonisospectral KP equation

$$4u_t[1] + y(u_{xxx}[1] + 6u[1]u_x[1] + 3\partial^{-1}u_{yy}[1]) + 2xu_y[1] + 4\partial^{-1}u_y[1] + 2u_x[1] = 0. \tag{3.2.12}$$

Proof: It is easy to find that (3.2.11a) hold. To prove (3.2.11b), we need to show the following equality:

$$\begin{aligned}
4\phi_t[1] &= 4(\phi_x - \tfrac{\phi_{1,x}}{\phi_1}\phi)_t \\
&= (yA(u)\phi + xB(u)\phi + C(u)\phi)_x - (\tfrac{yA(u)\phi_1 + xB(u)\phi_1 + C(u)\phi_1}{\phi_1})_x\phi \\
&\quad -\tfrac{\phi_{1,x}}{\phi_1}(yA(u)\phi + xB(u)\phi + C(u)\phi) \\
&= y[(A(u)\phi)_x - (\tfrac{A(u)\phi_1}{\phi_1})_x\phi - \tfrac{\phi_{1,x}}{\phi_1}A(u)\phi] \\
&\quad +x[(B(u)\phi)_x - (\tfrac{B(u)\phi_1}{\phi_1})_x\phi - \tfrac{\phi_{1,x}}{\phi_1}B(u)\phi] \\
&\quad +B(u)\phi - \tfrac{B(u)\phi_1}{\phi_1}\phi + (C(u)\phi)_x - (\tfrac{C(u)\phi_1}{\phi_1})_x\phi - \tfrac{\phi_{1,x}}{\phi_1}(C(u)\phi) \\
&= yA(u[1])\phi[1] + xB(u[1])\phi[1] + C(u[1])\phi[1] - 2\phi_x[1].
\end{aligned} \tag{3.2.13}$$

Lemma2.1 implies the coefficient of y hold. Therefore we only need to check the equality:

$$(B(u)\phi)_x - (\frac{B(u)\phi_1}{\phi_1})_x\phi - \frac{\phi_{1,x}}{\phi_1}B(u)\phi = B(u[1])\phi[1], \tag{3.2.14a}$$

$$B(u)\phi - \frac{B(u)\phi_1}{\phi_1}\phi + (C(u)\phi)_x - (\frac{C(u)\phi_1}{\phi_1})_x\phi - \frac{\phi_{1,x}}{\phi_1}C(u)\phi = C(u[1])\phi[1] - 2\phi_x[1]. \tag{3.2.14b}$$

Using (3.2.10a), we have

$$B(u[1])\phi[1] = -2(\partial^2 + u[1])(\phi_x - \frac{\phi_{1,x}}{\phi_1}\phi)$$

$$= 2(-\phi_{xxx} + \frac{\phi_{1,xx}}{\phi_1}\phi - \frac{\phi_{1,x}\phi_{1,xx}}{\phi_1^2}\phi + \frac{\phi_{1,x}}{\phi_1}\phi_{xx} - u\phi_x + u\frac{\phi_{1,x}}{\phi_1}\phi), \tag{3.2.15}$$

and the left terms in (3.2.14a)

$$= (-2\phi_{xx} - 2u\phi)_x - (\frac{-2\phi_{1,xx} - 2u\phi_1}{\phi_1})_x\phi - \frac{2\phi_{1,x}}{\phi_1}(-2\phi_{xx} - 2u\phi)$$

$$= -2\phi_{xxx} - 2u\phi_x + 2\frac{\phi_{1,xx}}{\phi_1}\phi + 2\frac{\phi_{1,x}}{\phi_1}\phi_{xx} + 2\frac{\phi_{1,x}}{\phi_1}u\phi - 2\frac{\phi_{1,x}\phi_{1,xx}}{\phi_1^2}\phi. \tag{3.2.16}$$

Comparing (3.2.15) with (3.2.16), it is immediately found that equality (3.2.14a) hold.

In the similar way, we can find that (3.2.14b) hold. Equations (3.2.11) lead to (3.2.12). This completes the proof.

Let $\phi_1, \phi_2, \cdots, \phi_n$ be solutions of (3.2.8). We define the Wronskian W of k functions $\phi_1, \phi_2, \cdots, \phi_k$ by

$$W(\phi_1, \cdots, \phi_k) = det(A), \quad A_{ij} = \frac{d^{j-1}}{dx^{j-1}}\phi_i, \quad 1 \le i, j \le k. \tag{3.2.17}$$

Using $u[i]$, $\phi[i]$ and $\phi_j[i]$ to denote the action of i-times repeated DT (3.2.10) on the initial solutions u, ϕ, ϕ_j. We have

$$\phi_{j,y}[i] = \phi_{j,xx}[i] + u[i]\phi_j[i], \tag{3.2.18a}$$

$$4\phi_{j,t}[i] = yA(u[i])\phi_j[i] + xB(u[i])\phi_j[i] + C(u[i])\phi_j[i] - 2i\phi_{j,x}[i]. \tag{3.2.18b}$$

Lemma 2.2: For arbitrary integral $l, k (1 \le l \le n-1, 1 \le k \le l-1)$, we have

$$W(\phi_{l+1}[l], \cdots, \phi_{l+k}[l]) = \frac{W(\phi_l[l-1], \phi_{l+1}[l-1], \cdots, \phi_{l+k}[l-1])}{\phi_l[l-1]}, \tag{3.2.19}$$

$$W(\phi_{l+1}[l], \cdots, \phi_{l+k}[l], \phi[l]) = \frac{W(\phi_l[l-1], \phi_{l+1}[l-1], \cdots, \phi_{l+k}[l-1], \phi[l-1])}{\phi_l[l-1]}. \tag{3.2.20}$$

Proof: According to (3.2.10a), we have

$$\phi_{l+i}[l] = \phi_{l+i,x}[l-1] - \frac{\phi_{l,x}[l-1]}{\phi_l[l-1]}\phi_{l+i}[l-1], \tag{3.2.21}$$

then

$$\begin{aligned}
\frac{\partial^{j-1}}{\partial x^{j-1}}\phi_{l+i}[l] &= \frac{\partial^{j-1}}{\partial x^{j-1}}(\phi_{l+i,x}[l-1] - \frac{\phi_{l,x}[l-1]}{\phi_l[l-1]}\phi_{l+i}[l-1]) \\
&= \frac{\partial^{j-1}}{\partial x^{j-1}}\phi_{l+i,x}[l-1] - \sum_{h=0}^{j-1} C_{j-1}^h (\frac{\phi_{l,x}[l-1]}{\phi_l[l-1]})^{(j-1-h)}\phi_{l+i}^{(h)}[l-1] \\
&= a_{i,j+1} - \sum_{h=0}^{j-1} C_{j-1}^h b_{0,j-h} a_{i,h+1} \\
&= a_{i,j+1} - (b_{0,j}a_{i,1} + C_{j-1}^1 b_{0,j-1}a_{i,2} + \cdots + C_{j-1}^{j-2} b_{0,2}a_{i,j-1} + b_{0,1}a_{i,j}),
\end{aligned} \tag{3.2.22}$$

where $b_{0,m} = (\frac{\phi_{l,x}[l-1]}{\phi_l[l-1]})^{(m)}$, $a_{i,n} = \frac{d^{n-1}}{dx^{n-1}}\phi_{l+i}[l-1]$, $m = 1,\cdots,j$, $n = 1,\cdots,j+1$, then

$$W(\phi_{l+1}[l],\cdots,\phi_{l+k}[l]) =$$

$$= \begin{vmatrix}
a_{1,2} - b_{0,1}a_{1,1} & a_{1,3} - (b_{0,2}a_{1,1} + b_{0,1}a_{1,2}) & \cdots a_{1,k+1} - (b_{0,k}a_{1,1} + C_{k-1}^1 b_{0,k-1}a_{1,2} + \cdots + b_{0,1}a_{1,k}) \\
a_{2,2} - b_{0,1}a_{2,1} & a_{2,3} - (b_{0,2}a_{2,1} + b_{0,1}a_{2,2}) & \cdots a_{2,k+1} - (b_{0,k}a_{2,1} + C_{k-1}^1 b_{0,k-1}a_{2,2} + \cdots + b_{0,1}a_{2,k}) \\
\cdots & \cdots & \cdots \\
a_{k,2} - b_{0,1}a_{k,1} & a_{k,3} - (b_{0,2}a_{k,1} + b_{0,1}a_{k,2}) & \cdots a_{k,k+1} - (b_{0,k}a_{k,1} + C_{k-1}^1 b_{0,k-1}a_{k,2} + \cdots + b_{0,1}a_{k,k})
\end{vmatrix}$$

$$= \begin{vmatrix}
a_{1,2} & a_{1,3} & \cdots & a_{1,k+1} \\
a_{2,2} & a_{2,3} & \cdots & a_{2,k+1} \\
\cdots & \cdots & \cdots & \\
a_{k,2} & a_{k,3} & \cdots & a_{k,k+1}
\end{vmatrix} + B_{1,2}\begin{vmatrix}
a_{1,1} & a_{1,3} & \cdots & a_{1,k+1} \\
a_{2,1} & a_{2,3} & \cdots & a_{2,k+1} \\
\cdots & \cdots & \cdots & \\
a_{k,1} & a_{k,3} & \cdots & a_{k,k+1}
\end{vmatrix} + B_{1,3}\begin{vmatrix}
a_{1,1} & a_{1,2} & a_{1,4} & \cdots & a_{1,k+1} \\
a_{2,1} & a_{2,2} & a_{2,4} & \cdots & a_{2,k+1} \\
\cdots & \cdots & \cdots & & \\
a_{k,1} & a_{k,2} & a_{k,4} & \cdots & a_{k,k+1}
\end{vmatrix}$$

$$+ \cdots + B_{1,k}\begin{vmatrix}
a_{1,1} & a_{1,k-1} & \cdots & a_{1,k+1} \\
a_{2,1} & a_{2,k-1} & \cdots & a_{2,k+1} \\
\cdots & \cdots & \cdots & \\
a_{k,1} & a_{k,k-1} & \cdots & a_{k,k+1}
\end{vmatrix} + B_{1,k+1}\begin{vmatrix}
a_{1,1} & a_{1,2} & \cdots & a_{1,k} \\
a_{2,1} & a_{2,2} & \cdots & a_{2,k} \\
\cdots & \cdots & \cdots & \\
a_{k,1} & a_{k,2} & \cdots & a_{k,k}
\end{vmatrix}$$

$$= \frac{1}{\phi_l[l-1]}\begin{vmatrix}
a_{0,1} & a_{0,2} & \cdots & a_{0,k+1} \\
a_{1,1} & a_{1,2} & \cdots & a_{1,k+1} \\
\cdots & \cdots & \cdots & \\
a_{k,1} & a_{k,2} & \cdots & a_{k,k+1}
\end{vmatrix} = \frac{W(\phi_l[l-1],\phi_{l+1}[l-1],\cdots,\phi_{l+k}[l-1])}{\phi_l[l-1]}, \tag{3.2.23}$$

where $B_{1,j} = (-1)^{j-1}\frac{\phi_l^{(j-1)}[l-1]}{\phi_l[l-1]}$ $(j = 2, \cdots, k+1)$.

Similarly, the formula (3.2.20) can be proved. This completes the proof.

Theorem 2.2: Assume that u is the solution of nonisospectral KP equation (3.2.7), $\phi_1, \phi_2, \cdots, \phi_N$ are the solutions of (3.2.8), then $N-$times repeated DT (3.2.10) is given by

$$\phi[N] = \frac{W(\phi_1, \phi_2, \cdots, \phi_N, \phi)}{W(\phi_1, \phi_2, \cdots, \phi_N)}, \tag{3.2.24}$$

$$u[N] = u + 2\partial^2 \ln W(\phi_1, \phi_2, \cdots, \phi_N), \tag{3.2.25}$$

and $u[N], \phi[N]$ satisfy

$$\phi_y[N] = \phi_{xx}[N] + u[N]\phi[N], \tag{3.2.26a}$$

$$4\phi_t[N] = yA(u[N])\phi[N] + xB(u[N])\phi[N] + C(u[N])\phi[N] - 2N\phi_x[N], \tag{3.2.26b}$$

and

$$4u_t[N] + y(u_{xxx}[N] + 6u[N]u_x[N] + 3\partial^{-1}u_{yy}[N]) + 2xu_y[N] + 4\partial^{-1}u_y[N] + 2Nu_x[N] = 0. \tag{3.2.27}$$

Proof: Using (3.2.10),(3.2.19) and (3.2.20)

$$\begin{aligned}\phi[N] &= \phi[N-1] - \frac{\phi_{N,x}[N-1]}{\phi_N[N-1]}\phi[N-1] = \frac{W(\phi_N[N-1], \phi[N-1])}{W(\phi_N[N-1])} \\ &= \frac{W(\phi_{N-1}[N-2], \phi_N[N-2], \phi[N-2])}{W(\phi_{N-1}[N-2], \phi_N[N-2])} = \cdots = \frac{W(\phi_1, \phi_2, \cdots, \phi_N, \phi)}{W(\phi_1, \phi_2, \cdots, \phi_N)}.\end{aligned} \tag{3.2.28}$$

$$\begin{aligned}u[N] &= u[N-1] + 2\partial^2 \ln \phi_N[N-1] \\ &= u[N-2] + 2\partial^2 \ln \phi_{N-1}[N-2] + 2\partial^2 \ln \frac{W(\phi_{N-1}[N-2], \phi_N[N-2])}{\phi_{N-1}[N-2]} \\ &= u[N-2] + 2\partial^2 \ln W(\phi_{N-1}[N-2], \phi_N[N-2]) = \cdots = u + 2\partial^2 \ln W(\phi_1, \phi_2, \cdots, \phi_N).\end{aligned} \tag{3.2.29}$$

It is easy to find (3.2.26) from the proposition 2.1.

For example, in order to find one-soliton slution for the nonisospectral KP equation (3.2.12), we start from the solution $u = 0$ for nonisospectral KP equation (3.2.7). The solution for (3.2.8) reads

$$\phi_1 = 1 + \omega_1(t)e^{k_1(t)x + k_1^2(t)y}, \quad k_{1,t}(t) = -2k_1^2(t), \quad \omega_{1,t}(t) = -\frac{1}{2}\omega_1(t)k_1(t). \tag{3.2.30a}$$

Then from (3.2.10b) we find that

$$u[1] = 2\partial^2 \ln(1 + \omega_1(t)e^{k_1(t)x + k_1^2(t)y + \xi_1^{(0)}}), \tag{3.2.31}$$

which is one-soliton solution for the nonisospetral KP equation (3.2.12).

3.3 Bilinear Bäcklund transformation for the nonisospectral KP equation

In this section, we first derive a BT in bilinear form of (3.2.7) from the Lax pair (3.2.8). Through the transformation

$$u = 2(\ln f)_{xx}, \quad \phi = \frac{g}{f}, \tag{3.3.1}$$

it is not difficult to derive the bilinear form as follows

$$D_y g \cdot f = D_x^2 g \cdot f, \tag{3.3.2a}$$

$$4D_t g \cdot f + y(D_x^3 g \cdot f + 3D_x D_y g \cdot f) + 2xD_x^2 g \cdot f + 2fg_x = 0. \tag{3.3.2b}$$

Substituting $g = 1$ into (3.3.2*a*) and (3.3.2*b*), we have

$$-f_y = f_{xx}, \tag{3.3.3a}$$

$$-4f_t + y(-f_{xxx} + 3f_{xy}) + 2xf_{xx} = 0, \tag{3.3.3b}$$

then

$$f = f_1 = e^{\xi_1} + e^{-\eta_1}, \tag{3.3.4a}$$

$$\xi_1 = k_1(t)x - k_1^2(t)y + \xi_1^{(0)}, \quad \eta_1 = -q_1(t)x - q_1^2(t)y + \eta_1^{(0)}, \quad k_{1,t}(t) = \frac{1}{2}k_1^2(t), \quad q_{1,t}(t) = -\frac{1}{2}q_1^2(t), \tag{3.3.4b}$$

$$k_1(t) = \frac{2}{2c_1 - t}, \quad q_1(t) = \frac{2}{2c_1 + t}. \tag{3.3.4c}$$

Thus we can derive one-soliton solution for the nonisospectral KP equation

$$u = 2[\ln(e^{\xi_1} + e^{-\eta_1})]_{xx}. \tag{3.3.5}$$

If we take $g = f_1$, we can not obtain the two-soliton solution for the nonisospectral KP equation(3.2.7). Taking

$$g = a_1(t)e^{\xi_1} + b_1(t)e^{-\eta_1}, \tag{3.3.6a}$$

$$a_1(t) = \frac{2}{k_1(t)}, \quad b_1(t) = \frac{2}{p_1(t)}, \tag{3.3.6b}$$

which is the solution to the equation

$$4u_t + y(u_{xxx} + 6uu_x + 3\partial^{-1}u_{yy}) + 2xu_y + 4\partial^{-1}u_y + 2u_x = 0, \quad u = 2(\log g)_{xx}, \tag{3.3.7}$$

then the solution generated by the bilinear BT (3.3.2) is given by

$$f = f_2 = c_1(t)e^{\xi_1+\xi_2} + c_2(t)e^{-\eta_1-\eta_2} + c_3(t)e^{\xi_1-\eta_2} + c_4(t)e^{-\eta_1+\xi_2}, \tag{3.3.8a}$$

$$c_1(t) = [k_1(t) - k_2(t)], \quad c_2(t) = [q_1(t) - q_2(t)],$$

$$c_3(t) = -[k_1(t) + q_2(t)], \quad c_4(t) = -[k_2(t) + q_1(t)], \tag{3.3.8b}$$

$$\xi_1 = k_1(t)x - k_1^2(t)y + \xi_1^{(0)}, \quad \eta_1 = -q_1(t)x - q_1^2(t)y + \eta_1^{(0)},$$

$$k_{1,t}(t) = \frac{1}{2}k_1^2(t), \quad k_{2,t}(t) = \frac{1}{2}k_2^2(t), \quad q_{1,t}(t) = -\frac{1}{2}q_1^2(t), \quad q_{2,t}(t) = -\frac{1}{2}q_2^2(t), \tag{3.3.8c}$$

$$a_2(t) = \frac{2}{k_2(t)}, \quad b_2(t) = \frac{2}{p_2(t)}. \tag{3.3.8d}$$

Similar to two-soliton solution, three-soliton solution can be derived.

Taking the solution of (3.3.7)

$$g = c_1(t)a_1(t)a_2(t)e^{\xi_1+\xi_2} + c_2(t)b_1(t)b_2(t)e^{-\eta_1-\eta_2}$$

$$+c_3(t)a_1(t)b_2(t)e^{\xi_1-\eta_2} + c_4(t)b_1(t)a_2(t)e^{-\eta_1+\xi_2}, \tag{3.3.9a}$$

$$\xi_j = k_j(t)x - k_j^2(t)y + \xi_j^{(0)}, \quad \eta_j = -q_j(t)x - q_j^2(t)y + \eta_j^{(0)},$$

$$k_{j,t}(t) = -\frac{1}{2}k_j^2(t), \quad q_{j,t}(t) = \frac{1}{2}q_j^2(t), \quad a_j(t) = (\frac{2}{k_j(t)})^2, \quad b_j(t) = (\frac{2}{p_j(t)})^2, \quad (j = 1,2), \tag{3.3.9b}$$

from (3.3.2), we can derive

$$f = f_3 c_5(t)a_1(t)a_2(t)a_3(t)e^{\xi_1+\xi_2+\xi_3} + c_6(t)a_1(t)a_2(t)b_3(t)e^{\xi_1+\xi_2-\eta_3}$$

$$+c_7(t)a_1(t)b_2(t)a_3(t)e^{\xi_1-\eta_2+\xi_3} + c_8(t)b_1(t)a_2(t)a_3(t)e^{-\eta_1+\xi_2+\xi_3}$$

$$+c_9(t)b_1(t)b_2(t)a_3(t)e^{-\eta_1-\eta_2+\xi_3} + c_{10}(t)b_1(t)a_2(t)b_3(t)e^{-\eta_1+\xi_2-\eta_3}$$

$$+c_{11}(t)a_1(t)b_2(t)b_3(t)e^{\xi_1-\eta_2-\eta_3} + c_{12}(t)b_1(t)b_2(t)b_3(t)e^{-\eta_1-\eta_2-\eta_3} \quad , \tag{3.3.10a}$$

$$c_5(t) = [k_1(t) - k_2(t)][k_2(t) - k_3(t)][k_1(t) - k_3(t)],$$

$$c_6(t) = [k_1(t) - k_2(t)][k_2(t) + q_3(t)][k_1(t) + q_3(t)],$$

$$c_7(t) = [k_1(t) + q_2(t)][k_3(t) + q_2(t)][k_1(t) - k_3(t)],$$

$$c_8(t) = [k_2(t) - k_3(t)][k_2(t) + q_1(t)][k_3(t) + q_1(t)],$$

$$c_9(t) = [k_3(t) + q_1(t)][k_3(t) + q_2(t)][q_1(t) - q_2(t)],$$

$$c_{10}(t) = [k_2(t) + q_1(t)][k_2(t) + q_3(t)][q_1(t) - q_3(t)],$$

$$c_{11}(t) = [k_1(t) + q_2(t)][q_2(t) - q_3(t)][k_1(t) + q_3(t)],$$

$$c_{12}(t) = [q_1(t) - q_2(t)][q_1(t) - q_3(t)][q_2(t) - q_3(t)], \tag{3.3.10b}$$

where

$$\xi_3 = k_3(t)x - k_3^2(t)y + \xi_3^{(0)}, \quad \eta_3 = -q_3(t)x - q_3^2(t)y + \eta_3^{(0)},$$

$$k_{3,t}(t) = \frac{1}{2}k_3^2(t), \quad q_{3,t}(t) = -\frac{1}{2}q_3^2(t), \quad a_3(t) = (\frac{2}{k_3(t)})^2, \quad b_3(t) = (\frac{2}{p_3(t)})^2. \tag{3.3.10c}$$

Generally, take the solution of (3.3.7)

$$g = \sum_{\epsilon=0,1} \prod_{1\le j<l}^{N-1} (2\epsilon_l - 1)[\epsilon_j k_j(t) + (\epsilon_j - 1)q_j(t) - \epsilon_l k_l(t) - (\epsilon_l - 1)q_l(t)]$$

$$\exp\{\sum_{j=1}^{N-1} [\epsilon_j(\xi_j + \ln a_j(t)) + (\epsilon_j - 1)(\eta_j - \ln b_j(t))]\}, \tag{3.3.11a}$$

here

$$\xi_j = k_j(t)x - k_j^2(t)y + \xi_j^{(0)}, \quad \eta_j = -q_j(t)x - q_j^2(t)y + \eta_j^{(0)}, \tag{3.3.11b}$$

$$k_{j,t}(t) = \frac{1}{2}k_j^2(t), \quad q_{j,t}(t) = -\frac{1}{2}q_j^2(t), \quad a_j(t) = \left(\frac{2}{k_j(t)}\right)^{N-1}, \quad b_j(t) = \left(\frac{2}{q_j(t)}\right)^{N-1}, \tag{3.3.11c}$$

we can get

$$f = f_N = \sum_{\epsilon=0,1} \prod_{1\le j<l}^{N} (2\epsilon_l - 1)[\epsilon_j k_j(t) + (\epsilon_j - 1)q_j(t) - \epsilon_l k_l(t) - (\epsilon_l - 1)q_l(t)]$$

$$\exp\{\sum_{j=1}^{N}[\epsilon_j(\xi_j + \ln a_j(t)) + (\epsilon_j - 1)(\eta_j - \ln b_j(t))]\}, \tag{3.3.12a}$$

$$\xi_N = k_N(t)x - k_N^2(t)y + \xi_N^{(0)}, \quad \eta_N = -q_N(t)x - q_N^2(t)y + \eta_N^{(0)}, \tag{3.3.12b}$$

$$k_{N,t}(t) = \frac{1}{2}k_N^2(t), \quad q_{N,t}(t) = -\frac{1}{2}q_N^2(t), \quad a_N(t) = \left(\frac{2}{k_N(t)}\right)^{N-1}, \quad b_N(t) = \left(\frac{2}{q_N(t)}\right)^{N-1}. \tag{3.3.12c}$$

Finally, we obtain the solution in Wronskian form for bilinear BT (3.3.2).

Let

$$f = W(\phi_1, \phi_2, \cdots, \phi_N) = \begin{vmatrix} \phi_1 & \partial\phi_1 & \cdots & \partial^{N-1}\phi_1 \\ \phi_2 & \partial\phi_2 & \cdots & \partial^{N-1}\phi_2 \\ \cdots & \cdots & \cdots & \cdots \\ \phi_N & \partial\phi_N & \cdots & \partial^{N-1}\phi_N \end{vmatrix} = |0, 1, \cdots, N-1| = |\widehat{N-1}|, \tag{3.3.13a}$$

$$g = |\widehat{N-2}, \tau|, \quad \tau = |0, \cdots, 0, 1|^T, \tag{3.3.13b}$$

where ϕ_j satisfies

$$\phi_{j,y} = -\phi_{j,xx}, \quad \phi_{j,t} = -y\phi_{j,xxx} + \frac{1}{2}x\phi_{j,xx} - \frac{1}{2}(N-1)\phi_{j,x}. \tag{3.3.14a, b}$$

From (3.3.14b), we have

$$\begin{aligned} g_t = &-y(|\widehat{N-5}, N-3, N-2, N-1, \tau| - |\widehat{N-4}, N-2, N, \tau| + |\widehat{N-3}, N+1, \tau|) \\ &-\frac{1}{2}x|\widehat{N-4}, N-2, N-1, \tau| + \frac{1}{2}x|\widehat{N-3}, N, \tau| - \frac{1}{2}|\widehat{N-3}, N-1, \tau|, \end{aligned} \tag{3.3.15}$$

$$\begin{aligned} f_t = &-y(|\widehat{N-4}, N-2, N-1, N| - |\widehat{N-3}, N-1, N+1| + |\widehat{N-2}, N+2|) \\ &-\frac{1}{2}x|\widehat{N-3}, N-1, N| + \frac{1}{2}x|\widehat{N-2}, N+1|. \end{aligned} \tag{3.3.16}$$

Then it is easy to prove that (3.3.13) with (3.3.14) solve (3.3.2) in the way similar to the Ref.[12].

Define the entries

$$\phi_j = a_j(t)e^{\xi_j} + b_j(t)e^{-\eta_j}, \quad \xi_j = k_j(t)x - k_j^2(t)y + \xi_j^{(0)}, \quad \eta_j = -q_j(t)x - q_j^2(t)y + \eta_j^{(0)}, \tag{3.3.17a}$$

$$k_{j,t}(t) = \frac{1}{2}k_j^2(t), \quad q_{j,t}(t) = -\frac{1}{2}q_j^2(t), \quad a_j(t) = \left(\frac{2}{k_j(t)}\right)^{N-1}, \quad b_j(t) = \left(\frac{2}{q_j(t)}\right)^{N-1}. \tag{3.3.17b}$$

Similar to Ref.[14], the Wronskian (3.3.13a) can be written as

$$\begin{aligned} f = (-1)^{\frac{N(N-1)}{2}} \sum_{\epsilon=0,1} \prod_{1\le j<l}^{N} (2\epsilon_l - 1)[\epsilon_j k_j(t) + (\epsilon_j - 1)q_j(t) - \epsilon_l k_l(t) - (\epsilon_l - 1)q_l(t)] \\ \exp\{\sum_{j=1}^{N}[\epsilon_j(\xi_j + \ln a_j(t)) + (\epsilon_j - 1)(\eta_j - b_j(t))]\}. \end{aligned} \tag{3.3.18}$$

which is the same to (3.3.12).

Chapter 4
The Multisoliton Solutions of the mKP Equation with Self-consistent Sources

The mKP equation with self-consistent sources is derived through the linear problem of the mKP system. The bilinear form of the mKP equation with self-consistent sources is given and the N-soliton solutions are obtained through Hirota method and Wronskian technique respectively. The coincidence of these solutions is shown by direct computation.

4.1 Introduction

The study of the soliton equations with self-consistent sources (SESCS) have received considerable attention in recent years. The reason may be that these equations can model many physical interesting processes and also result many mathematical interesting treatments[21-36]. The SESCS can be constructed through some mathematical ways[24-29]. One of simple methods is using the high-order constrained flows of soliton equations, namely, the high-order constrained flows of soliton equations are considered as the stationary equations of the SESCS[27-29]. Most of the sources obtained in this way are all related to eigenfunctions because the variational derivatives of eigenvalues are related to eigenfunctions. Some studies have also shown the SESCS exhibit multisoliton solutions[27],[30-36]. With the help of some special treatments, the inverse scattering method and Darboux transformation have been successfully used to find N-soliton solutions of the SESCS such as the KdV, KP and mKP equations with self-consistent sources. It will be shown from the solutions that the sources may result the variation of the velocity of solitons[25],[37].

One of the purposes of this paper is to derive the hierarchy of the mKP equation with self-consistent sources in the way which is directly based on the eigenfunctions of recursion operator. This method, different slightly from the one using constraint flows, is easy to give the Lax representations of the hierarchy. We have found that some other hierarchies of the SESCS, such as the KP equation with self-consistent sources[14], can be also obtained in this way. On the other hand, we also hope to find the multi-soliton solutions of the mKP equation with self-consistent sources (mKPESCS) through Hirota method[10] and Wronskian technique[38-41]. These two direct methods both depend on the bilinear forms of the evolution equations. Hirota method provides a remarkably simpler technique for obtaining the N-soliton solutions in the form of an Nth-order polynomial in N exponentials. Wronskian technique provides an alternative formulation of the N-soliton solutions, in terms of some function of the Wronski determinant of N functions, which allows verification of the solutions by direct substitution because differentiation of a Wronskian is easy and its derivatives take similar compact forms. The basic thoughts of our obtaining the exact N-soliton solutions are as follows. We first present a set of dependent variable transformations to write out the bilinear form of the mKPESCS by which we can derive one-, two-, even three-soliton solutions successively through the standard Hirota's approach. These results can help us to find out the time evolution easily and conjecture a general formula which denotes N-soliton solution but is only conjectured and not verified. Next, with the help of the message on the time evolution obtained by means of Hirota method, we can construct a Wronskian and try to verify it to satisfy the related bilinear equations. Since there is a nonlinear term (led to by the concerned source) in the time evolution, we have to develop some novel determinantal identities and employ some special treatments which are different from the known standard Wronskian technique[38-41] so that we can finish the Wronskian verifications. Finally, we present a process to show that the solutions of the bilinear equations obtained through the above two direct methods are the same for recovering the solutions of mKPESCS from the original dependent variable transformations. In other words, these two kinds of solutions are uniform. To our knowledge, it is the first time to obtain the mKPESCS and solve it by Hirota method and Wronskian technique.

We arrange the paper as follows. We first derive the hierarchy of of the mKPESCS in Sec.2. Then we solve the mKPESCS by means of Hirota method and Wronskian technique in Sec.3 and 4 respectively. At last, in Sec.5 we show the uniformity of the results in Sec.3 and 4.

4.2 The mKP equation with self-consistent sources

Consider the spectral problem and its adjoint associated with the mKP equation

$$\Phi_y = \Phi_{xx} + 2u\Phi_x, \tag{4.2.1}$$

$$\Psi_y = -\Psi_{xx} + 2u\Psi_x. \tag{4.2.2}$$

Suppose that the time evolution of the eigenfunction Φ is given by

$$\Phi_t = A\Phi, \tag{4.2.3}$$

where A is a operator function of ∂ and ∂^{-1} ($\partial = \frac{\partial}{\partial x}$ and $\partial^{-1}\partial = \partial\partial^{-1} = 1$). The compatibility of (4.2.1) and (4.2.3) requires that A satisfy

$$2u_t\partial - A_y + [\partial^2 + 2u\partial, A] = 0, \tag{4.2.4}$$

or

$$2u_t\partial = A_y - A_{xx} - 2A_x\partial - 2uA_x - 2[u, A]\partial. \tag{4.2.5}$$

Now we take

$$A = a_0\partial^3 + a_1\partial^2 + a_2\partial + \alpha(\Phi\Psi - \Phi\partial^{-1}\Psi_x), \tag{4.2.6}$$

where $a_j (j = 0, 1, 2)$ are undetermined functions of u and its derivatives, and α is an arbitrary constant. Substituting (4.2.6) into (4.2.5) and equating coefficients powers of ∂, we obtain

$$2u_t = a_{2,y} - a_{2,xx} - 2ua_{2,x} + 2a_0u_{xxx} + 2a_1u_{xx} + 2a_2u_x - 2\alpha(\Phi\Psi)_x, \tag{4.2.7}$$

$$a_{1,y} - a_{1,xx} - 2a_{2,x} - 2ua_{1,x} + 6a_0u_{xx} + 4a_1u_x = 0, \tag{4.2.8}$$

$$a_{0,y} - a_{0,xx} - 2a_{1,x} - 2ua_{0,x} + 6a_0u_x = 0, \tag{4.2.9}$$

$$a_{0,x} = 0. \tag{4.2.10}$$

From (4.2.8)-(4.2.10), we work out in regular order that

$$a_0 = -4, \quad a_1 = -12u, \quad a_2 = -6\partial^{-1}u_y - 6u_x - 6u^2. \tag{4.2.11}$$

Substituting (4.2.11) into (4.2.7) and setting $\alpha = -1$, we obtain

$$u_t + u_{xxx} + 3\partial^{-1}u_{yy} - 6u^2u_x + 6(\partial^{-1}u_y)u_x - (\Phi\Psi)_x = 0. \tag{4.2.12}$$

This equation together with spetral problems (4.2.1) and (4.2.2) constitutes the mKP equation with a self-consistent source. If taking $\alpha = 0$, we can derive the mKP equation

$$u_t + u_{xxx} + 3\partial^{-1}u_{yy} - 6u^2u_x = 0. \tag{4.2.13}$$

Obviously, the mKP equation with N self-consistent sources can be defined in a similar way, which is expressed as

$$u_t + u_{xxx} + 3\partial^{-1}u_{yy} - 6u^2u_x + 6(\partial^{-1}u_y)u_x - \sum_{j=1}^{N}(\Phi_j\Psi_j)_x = 0, \tag{4.2.14}$$

$$\Phi_{j,y} = \Phi_{j,xx} + 2u\Phi_{j,x}, \tag{4.2.15}$$

$$\Psi_{j,y} = -\Psi_{j,xx} + 2u\Psi_{j,x}, \tag{4.2.16}$$

while the operator A becomes

$$A = -4\partial^3 - 12u\partial^2 - (6\partial^{-1}u_y + 6u_x + 6u^2)\partial - \sum_{j=1}^{N}(\Phi_j\Psi_j - \Phi_j\partial^{-1}\Psi_{j,x}). \tag{4.2.17}$$

4.3 Bilinear form and Hirota method

In the following, we shall give the soliton solution of the mKPESCS by use of Hirota method.

With the help of the dependent variable transformations

$$u = (\ln\frac{g}{f})_x, \quad \Phi_j = \frac{h_j}{g}, \quad \Psi_j = \frac{s_j}{f}, \tag{4.3.1}$$

the mKPESCS (4.2.14)-(4.2.16) can be transformed into the bilinear forms

$$D_x^2 g \cdot f - D_y g \cdot f = 0, \tag{4.3.2}$$

$$D_t g \cdot f + D_x^3 g \cdot f + 3D_x D_y g \cdot f = \sum_{j=1}^{N} h_j s_j, \tag{4.3.3}$$

$$D_y h_j \cdot f - D_x^2 h_j \cdot f = 0, \tag{4.3.4}$$

$$D_y s_j \cdot g + D_x^2 s_j \cdot g = 0, \tag{4.3.5}$$

where D is the well-known Hirota bilinear operator

$$D_x^l D_y^m D_t^n a \cdot b = (\partial_x - \partial_{x'})^l (\partial_y - \partial_{y'})^m (\partial_t - \partial_{t'})^n a(x,y,t) b(x',y',t')|_{x'=x,y'=y,t'=t}.$$

Expanding f, g and h_j, s_j as the series

$$f = 1 + f^{(2)}\epsilon^2 + f^{(4)}\epsilon^4 + f^{(6)}\epsilon^6 + \cdots, \tag{4.3.6}$$

$$g = 1 + g^{(2)}\epsilon^2 + g^{(4)}\epsilon^4 + \cdots, \tag{4.3.7}$$

$$h_j = h_j^{(1)}\epsilon + h_j^{(3)}\epsilon^3 + \cdots, \tag{4.3.8}$$

$$s_j = s_j^{(1)}\epsilon + s_j^{(3)}\epsilon^3 + \cdots. \tag{4.3.9}$$

Substituting (4.3.6)-(4.3.9) into (4.3.2)-(4.3.5) and equating coefficients of ϵ yield

$$g_{xx}^{(2)} - g_y^{(2)} + f_{xx}^{(2)} + f_y^{(2)} = 0, \tag{4.3.10}$$

$$g_{xx}^{(4)} - g_y^{(4)} + f_{xx}^{(4)} + f_y^{(4)} + D_x^2 g^{(2)} \cdot f^{(2)} - D_y g^{(2)} \cdot f^{(2)} = 0, \tag{4.3.11}$$

$$\cdots\cdots,$$

$$g_t^{(2)} + g_{xxx}^{(2)} + 3g_{xy}^{(2)} - f_t^{(2)} - f_{xxx}^{(2)} + 3f_{xy}^{(2)} = \sum_{j=1}^{N} h_j^{(1)} s_j^{(1)}, \tag{4.3.12}$$

$$g_t^{(4)} + g_{xxx}^{(4)} + 3g_{xy}^{(4)} - f_t^{(4)} - f_{xxx}^{(4)} + 3f_{xy}^{(4)} + D_t g^{(2)} \cdot f^{(2)} + D_x^2 g^{(2)} \cdot f^{(2)}$$
$$+3D_x D_y g^{(2)} \cdot f^{(2)} = \sum_{j=1}^{N} (h_j^{(1)} s_j^{(3)} + h_j^{(3)} s_j^{(1)}), \tag{4.3.13}$$

$$\cdots\cdots,$$

$$h_{j,y}^{(1)} - h_{j,xx}^{(1)} = 0, \tag{4.3.14}$$

$$h_{j,y}^{(3)} - h_{j,xx}^{(3)} + D_y h_j^{(1)} \cdot f^{(2)} - D_x^2 h_j^{(1)} \cdot f^{(2)} = 0, \tag{4.3.15}$$

$$\cdots\cdots.$$

$$s_{j,y}^{(1)} + s_{j,xx}^{(1)} = 0, \tag{4.3.16}$$

$$s_{j,y}^{(3)} + s_{j,xx}^{(3)} + D_y s_j^{(1)} \cdot g^{(2)} + D_x^2 h_j^{(1)} \cdot f^{(2)} = 0, \tag{4.3.17}$$

$$\cdots\cdots.$$

For $N = 1$, let

$$h_1^{(1)} = -\sqrt{2(k_1 + q_1)\beta_1(t)} e^{\xi_1}, \quad \xi_1 = k_1 x + k_1^2 y - 4k_1^3 t - \int_0^t \beta_1(z) dz + \xi_1^{(0)}, \tag{4.3.18}$$

$$s_1^{(1)} = \sqrt{2(k_1 + q_1)\beta_1(t)} e^{\eta_1}, \quad \eta_1 = q_1 x - q_1^2 y - 4q_1^3 t - \int_0^t \beta_1(z) dz + \eta_1^{(0)}. \tag{4.3.19}$$

By solving (4.3.10)-(4.3.17), we have

$$f^{(2)} = b_1 e^{\xi_1 + \eta_1}, \quad b_1 = -q_1, \tag{4.3.20}$$

$$g^{(2)} = a_1 e^{\xi_1 + \eta_1}, \quad a_1 = k_1, \tag{4.3.21}$$

$$h_1^{(l)} = 0, \quad s_1^{(l)} = 0, \quad l = 3, 5, \cdots, \tag{4.3.22}$$

$$f^{(m)} = 0, \quad g^{(m)} = 0, \quad m = 4, 6, \cdots, \tag{4.3.23}$$

therefore the one-soliton solution is given by

$$u = [\ln \frac{1 + a_1 e^{\xi_1 + \eta_1}}{1 + b_1 e^{\xi_1 + \eta_1}}]_x, \tag{4.3.24}$$

$$\Phi_1 = \frac{-\sqrt{2(k_1+q_1)\beta_1(t)}e^{\xi_1}}{1+a_1e^{\xi_1+\eta_1}}, \quad \Psi_1 = \frac{\sqrt{2(k_1+q_1)\beta_1(t)}e^{\eta_1}}{1+b_1e^{\xi_1+\eta_1}}. \tag{4.3.25}$$

For $N=2$, if we take

$$h_j^{(1)} = -\sqrt{2(k_j+q_j)\beta_j(t)}e^{\xi_j}, \quad \xi_j = k_jx + k_j^2y - 4k_j^3t - \int_0^t \beta_j(z)dz + \xi_j^{(0)}, \tag{4.3.26}$$

$$s_j^{(1)} = \sqrt{2(k_j+q_j)\beta_j(t)}e^{\eta_j}, \quad \eta_j = q_jx - q_j^2y - 4q_j^3t - \int_0^t \beta_j(z)dz + \eta_j^{(0)}, \quad j=1,2. \tag{4.3.27}$$

From (4.3.10)-(4.3.17) it can be worked out that

$$f^{(2)} = b_1e^{\xi_1+\eta_1} + b_2e^{\xi_2+\eta_2}, \quad b_j = -q_j, \tag{4.3.28}$$

$$g^{(2)} = a_1e^{\xi_1+\eta_1} + a_2e^{\xi_2+\eta_2}, \quad a_j = k_j, \tag{4.3.29}$$

$$h_1^{(3)} = -\sqrt{2(k_1+q_1)\beta_1(t)}b_2\frac{(k_2-k_1)}{(k_1+q_2)}e^{\xi_1+\xi_2+\eta_2+i\pi}, \tag{4.3.30}$$

$$h_2^{(3)} = -\sqrt{2(k_2+q_2)\beta_2(t)}b_1\frac{(k_2-k_1)}{(k_2+q_1)}e^{\xi_2+\xi_1+\eta_1}, \tag{4.3.31}$$

$$s_1^{(3)} = \sqrt{2(k_1+q_1)\beta_1(t)}a_2\frac{(q_2-q_1)}{(q_1+k_2)}e^{\eta_1+\xi_2+\eta_2+i\pi}, \tag{4.3.32}$$

$$s_2^{(3)} = \sqrt{2(k_2+q_2)\beta_2(t)}a_1\frac{(q_2-q_1)}{(q_2+k_1)}e^{\eta_2+\xi_1+\eta_1}, \tag{4.3.33}$$

$$f^{(4)} = b_1b_2e^{\xi_1+\eta_1+\xi_2+\eta_2+A_{12}}, \tag{4.3.34}$$

$$g^{(4)} = a_1a_2e^{\xi_1+\eta_1+\xi_2+\eta_2+A_{12}}, \tag{4.3.35}$$

$$e^{A_{12}} = \frac{(k_1-k_2)(q_1-q_2)}{(k_1+q_2)(k_2+q_1)}, \tag{4.3.36}$$

$$h_j^{(l)} = 0, \quad s_j^{(l)} = 0, \quad j=1,2, \quad l=3,5, \tag{4.3.37}$$

$$f^{(m)} = 0, \quad g^{(m)} = 0, \quad m = 6, 8, \cdots. \tag{4.3.38}$$

So the two-soliton solution is

$$u = [\frac{1+a_1e^{\xi_1+\eta_1}+a_2e^{\xi_2+\eta_2}+a_1a_2e^{\xi_1+\eta_1+\xi_2+\eta_2+A_{12}}}{1+b_1e^{\xi_1+\eta_1}+b_2e^{\xi_2+\eta_2}+b_1b_2e^{\xi_1+\eta_1+\xi_2+\eta_2+A_{12}}}]_x, \tag{4.3.39}$$

$$\Phi_1 = -\sqrt{2(k_1+q_1)\beta_1(t)}e^{\xi_1}\frac{1+b_2\frac{(k_2-k_1)}{(k_1+q_2)}e^{\xi_1+\xi_2+\eta_2+i\pi}}{1+a_1e^{\xi_1+\eta_1}+a_2e^{\xi_2+\eta_2}+a_1a_2e^{\xi_1+\eta_1+\xi_2+\eta_2+A_{12}}}, \tag{4.3.40}$$

$$\Phi_2 = -\sqrt{2(k_2+q_2)\beta_2(t)}e^{\xi_2}\frac{1+b_1\frac{(k_2-k_1)}{(k_2+q_1)}e^{\xi_2+\xi_1+\eta_1}}{1+a_1e^{\xi_1+\eta_1}+a_2e^{\xi_2+\eta_2}+a_1a_2e^{\xi_1+\eta_1+\xi_2+\eta_2+A_{12}}}, \tag{4.3.41}$$

$$\Psi_1 = \sqrt{2(k_1+q_1)\beta_1(t)}e^{\eta_1}\frac{1+a_2\frac{(q_2-q_1)}{(q_1+k_2)}e^{\eta_1+\xi_2+\eta_2+i\pi}}{1+b_1e^{\xi_1+\eta_1}+b_2e^{\xi_2+\eta_2}+b_1b_2e^{\xi_1+\eta_1+\xi_2+\eta_2+A_{12}}}, \tag{4.3.42}$$

$$\Psi_2 = \sqrt{2(k_2+q_2)\beta_2(t)}e^{\eta_2}\frac{a_1\frac{(q_2-q_1)}{(q_2+k_1)}e^{\eta_2+\xi_1+\eta_1}}{1+b_1e^{\xi_1+\eta_1}+b_2e^{\xi_2+\eta_2}+b_1b_2e^{\xi_1+\eta_1+\xi_2+\eta_2+A_{12}}}. \tag{4.3.43}$$

Similar, for $N = 3$ the three soliton solution can be derived, where

$$\begin{aligned} g = 1 &+ a_1e^{\xi_1+\eta_1}+a_2e^{\xi_2+\eta_2}+a_3e^{\xi_3+\eta_3}+a_1a_2e^{\xi_1+\eta_1+\xi_2+\eta_2+A_{12}} \\ &+a_1a_3e^{\xi_1+\eta_1+\xi_3+\eta_3+A_{13}}+a_2a_3e^{\xi_2+\eta_2+\xi_3+\eta_3+A_{13}} \\ &+a_1a_2a_3e^{\xi_1+\eta_1+\xi_2+\eta_2+\xi_3+\eta_3+A_{12}+A_{13}+A_{23}}, \end{aligned} \tag{4.3.44}$$

$$\begin{aligned} f = 1 &+ b_1e^{\xi_1+\eta_1}+b_2e^{\xi_2+\eta_2}+b_3e^{\xi_3+\eta_3}+b_1b_2e^{\xi_1+\eta_1+\xi_2+\eta_2+A_{12}} \\ &+b_1b_3e^{\xi_1+\eta_1+\xi_3+\eta_3+A_{13}}+b_2b_3e^{\xi_2+\eta_2+\xi_3+\eta_3+A_{13}} \\ &+b_1b_2b_3e^{\xi_1+\eta_1+\xi_2+\eta_2+\xi_3+\eta_3+A_{12}+A_{13}+A_{23}}, \end{aligned} \tag{4.3.45}$$

$$e^{A_{jl}} = \frac{(k_j-k_l)(q_j-q_l)}{(k_j+q_l)(k_l+q_j)}, \quad (j<l, j, l = 1,2,3). \tag{4.3.46}$$

$$\begin{aligned} h_1 = -\sqrt{2(k_1+q_1)\beta_1(t)}e^{\xi_1}[&b_2e^{\xi_2+\eta_2+i\pi}\frac{(k_2-k_1)}{(k_1+q_2)}+b_3e^{\xi_3+\eta_3+i\pi}\frac{(k_3-k_1)}{(k_1+q_3)} \\ &+b_2b_3e^{\xi_2+\eta_2+\xi_3+\eta_3+A_{23}}\frac{(k_2-k_1)}{(k_1+q_2)}\frac{(k_3-k_1)}{(k_1+q_3)}], \end{aligned} \tag{4.3.47}$$

$$\begin{aligned} h_2 = -\sqrt{2(k_2+q_2)\beta_2(t)}e^{\xi_2}[&b_1e^{\xi_1+\eta_1}\frac{(k_2-k_1)}{(k_2+q_1)}+b_3e^{\xi_3+\eta_3+i\pi}\frac{(k_3-k_2)}{(k_2+q_3)} \\ &+b_1b_3e^{\xi_1+\eta_1+\xi_3+\eta_3+i\pi+A_{13}}\frac{(k_2-k_1)}{(k_2+q_1)}\frac{(k_3-k_2)}{(k_2+q_3)}], \end{aligned} \tag{4.3.48}$$

$$\begin{aligned} h_3 = -\sqrt{2(k_3+q_3)\beta_3(t)}e^{\xi_3}[&b_1e^{\xi_1+\eta_1}\frac{(k_3-k_1)}{(k_3+q_1)}+b_2e^{\xi_2+\eta_2}\frac{(k_3-k_2)}{(k_3+q_2)} \\ &+b_1b_2e^{\xi_1+\eta_1+\xi_2+\eta_2+A_{12}}\frac{(k_3-k_1)}{(k_3+q_1)}\frac{(k_3-k_2)}{(k_3+q_2)}], \end{aligned} \tag{4.3.49}$$

$$\begin{aligned} s_1 = \sqrt{2(k_1+q_1)\beta_1(t)}e^{\eta_1}[&b_2e^{\xi_2+\eta_2+i\pi}\frac{(q_2-q_1)}{(q_1+k_2)}+b_3e^{\xi_3+\eta_3+i\pi}\frac{(q_3-q_1)}{(q_1+k_3)} \\ &+b_2b_3e^{\xi_2+\eta_2+\xi_3+\eta_3+A_{23}}\frac{(q_2-q_1)}{(q_1+k_2)}\frac{(q_3-q_1)}{(q_1+k_3)}], \end{aligned} \tag{4.3.50}$$

$$\begin{aligned} s_2 = \sqrt{2(k_2+q_2)\beta_2(t)}e^{\eta_2}[&b_1e^{\xi_1+\eta_1}\frac{(q_2-q_1)}{(q_2+k_1)}+b_3e^{\xi_3+\eta_3+i\pi}\frac{(q_3-q_2)}{(q_2+k_3)} \\ &+b_1b_3e^{\xi_1+\eta_1+\xi_3+\eta_3+i\pi+A_{13}}\frac{(q_2-q_1)}{(q_2+k_1)}\frac{(q_3-q_2)}{(q_2+k_3)}], \end{aligned} \tag{4.3.51}$$

$$s_3 = \sqrt{2(k_3+q_3)\beta_3(t)}e^{\eta_3}[b_1e^{\xi_1+\eta_1}\frac{(q_3-q_1)}{(q_3+k_1)} + b_2e^{\xi_2+\eta_2}\frac{(q_3-q_2)}{(q_3+k_2)}$$

$$+b_1b_2e^{\xi_1+\eta_1+\xi_2+\eta_2+A_{12}}\frac{(q_3-q_1)}{(q_3+k_1)}\frac{(q_3-q_2)}{(q_3+k_2)}], \tag{4.3.52}$$

where

$$\xi_j = k_jx + k_j^2y - 4k_j^3t - \int_0^t \beta_j(z)dz + \xi_j^{(0)},$$

$$\eta_j = q_jx - q_j^2y - 4q_j^3t - \int_0^t \beta_j(z)dz + \eta_j^{(0)},$$

$$a_j = k_j, \quad b_j = -q_j, \quad j = 1,2,3. \tag{4.3.53}$$

Generally, we have

$$g = \sum_{\mu=0,1} \exp[\sum_{j=1}^{N} \mu_j(\xi_j + \eta_j + \alpha_j) + \sum_{1\le j<l}^{N} \mu_j\mu_l A_{jl}], \tag{4.3.54}$$

$$f = \sum_{\mu=0,1} \exp[\sum_{j=1}^{N} \mu_j(\xi_j + \eta_j + \gamma_j) + \sum_{1\le j<l}^{N} \mu_j\mu_l A_{jl}], \tag{4.3.55}$$

$$h_m = -2\sqrt{2(k_m+q_m)\beta_m(t)}e^{\xi_m} \sum_{\mu=0,1} \exp[\sum_{1\le j<m} \mu_j(\xi_j+\eta_j+\gamma_j+B_{mj})]$$

$$\exp[\sum_{j>m}^{N} \mu_j(\xi_j+\eta_j+\gamma_j+i\pi+B_{jm}) + \sum_{1\le j<l,j,l\ne m}^{N} \mu_j\mu_l A_{jl}], \tag{4.3.56}$$

$$s_m = 2\sqrt{2(k_m+q_m)\beta_m(t)}e^{\eta_m} \sum_{\mu=0,1} \exp[\sum_{1\le j<m} \mu_j(\xi_j+\eta_j+\alpha_j+C_{mj})]$$

$$\exp[\sum_{j>m}^{N} \mu_j(\xi_j+\eta_j+\alpha_j+i\pi+C_{jm}) + \sum_{1\le j<l,j,l\ne m}^{N} \mu_j\mu_l A_{jl}], \tag{4.3.57}$$

$$\xi_j = k_jx + k_j^2y - 4k_j^3t - \int_0^t \beta_j(z)dz + \xi_j^{(0)}, \tag{4.3.58}$$

$$\eta_j = q_jx - q_j^2y - 4q_j^3t - \int_0^t \beta_j(z)dz + \xi_j^{(0)}, \tag{4.3.59}$$

$$e^{A_{jl}} = \frac{(k_j-k_l)(q_j-q_l)}{(k_j+q_l)(k_l+q_j)}, \quad e^{B_{j1}} = \left(\frac{k_j-k_1}{k_1+q_j}\right). \quad e^{C_{j1}} = \left(\frac{q_j-q_1}{q_1+k_j}\right),$$

$$e^{B_{mj}} = \left(\frac{k_m-k_j}{k_m+q_j}\right), \quad e^{C_{mj}} = \left(\frac{q_m-q_j}{q_m+k_j}\right),$$

$$e^{\alpha_j} = a_j = k_j, \quad e^{\gamma_j} = b_j = -q_j. \tag{4.3.60}$$

here the sum is taken over all possible combinations of $\mu_j = 0,1$ $(j = 1,2,\cdots,N)$. When $\beta_j(t) = 0$, (4.3.54) and (4.3.55) is just the solution for mKP equation (4.2.13)[42].

4.4 Wronskian method

4.4.1 Wronskian method for the mKP equation

Through the transformation $u = (\ln \frac{g}{f})_x$, the bilinear form of the mKP equation is

$$D_x^2 g \cdot f - D_y g \cdot f = 0, \tag{4.4.1}$$

$$D_t g \cdot f + D_x^3 g \cdot f + 3D_x D_y g \cdot f = 0. \tag{4.4.2}$$

The mKP equation has the Wronskian form solutions as follows

$$f = \begin{vmatrix} \phi_1 & \partial\phi_1 & \cdots & \partial^{N-1}\phi_1 \\ \phi_2 & \partial\phi_2 & \cdots & \partial^{N-1}\phi_2 \\ \cdots & \cdots & \cdots & \cdots \\ \phi_N & \partial\phi_N & \cdots & \partial^{N-1}\phi_N \end{vmatrix} = |\phi, \phi^{(1)}, \cdots, \phi^{(N-1)}|$$

$$= |0, 1, \cdots, N-1| = |\widehat{N-1}|, \tag{4.4.3}$$

$$g = \begin{vmatrix} \partial\phi_1 & \partial^2\phi_1 & \cdots & \partial^{N}\phi_1 \\ \partial\phi_2 & \partial^2\phi_2 & \cdots & \partial^{N}\phi_2 \\ \cdots & \cdots & \cdots & \cdots \\ \partial\phi_N & \partial^2\phi_N & \cdots & \partial^{N}\phi_N \end{vmatrix} = |\phi^{(1)}, \phi^{(2)}, \cdots, \phi^{(N)}| = |1, 2, \cdots, N| = |\widetilde{N}|, \tag{4.4.4}$$

where ϕ_j satisfy

$$\phi_{j,y} = \phi_{j,xx}, \tag{4.4.5}$$

$$\phi_{j,t} = -4\phi_{j,xxx}. \tag{4.4.6}$$

From (4.4.5) and (4.4.6), it is easy to obtain

$$f_x = |\widehat{N-2}, N|, \quad f_{xx} = |\widehat{N-3}, N-1, N| + |\widehat{N-2}, N+1|, \tag{4.4.7}$$

$$f_{xx} = |\widehat{N-4}, N-2, N-1, N| + 2|\widehat{N-3}, N-1, N+1| + |\widehat{N-2}, N+2|, \tag{4.4.8}$$

$$f_y = -|\widehat{N-3}, N-1, N| + |\widehat{N-2}, N+1|, \tag{4.4.9}$$

$$f_{xy} = -|\widehat{N-4}, N-2, N-1, N| + |\widehat{N-2}, N+2|, \tag{4.4.10}$$

$$f_t = -4[|\widehat{N-4}, N-2, N-1, N| - |\widehat{N-3}, N-1, N+1| + |\widehat{N-2}, N+2|], \tag{4.4.11}$$

$$g_x = |\widetilde{N-1}, N+1|, \quad g_{xx} = |\widetilde{N-2}, N, N+1| + |\widetilde{N-1}, N+2|, \tag{4.4.12}$$

$$g_{xxx} = |\widetilde{N-3}, N-1, N, N+1| + 2|\widetilde{N-2}, n, N+2| + |\widetilde{N-1}, N+3|, \tag{4.4.13}$$

$$g_y=-|\widetilde{N-2},N,N-1|+|\widetilde{N-1},N+2|, \tag{4.4.14}$$

$$g_{xy}=-|\widetilde{N-3},N-1,N,N+1|+|\widetilde{N-1},N+3|, \tag{4.4.15}$$

$$g_t=-4[|\widetilde{N-3},N-1,N,N+1|-|\widetilde{N-2},N,N+2|+|\widetilde{N-1},N+3|, \tag{4.4.16}$$

Substituting (4.4.7)-(4.4.16) into (4.4.1)-(4.4.2), we have

$$\begin{aligned}
D_x^2g\cdot f-D_yg\cdot f&=g_{xx}f-2g_xf_x+gf_{xx}-(g_yf-gf_y)\\
&=(g_{xx}-g_y)f-2g_xf_x+g(f_{xx}+f_y)\\
=2|\widetilde{N-2},N,N+1||0,\widetilde{N-2},N-1|&-2|\widetilde{N-2},N-1,N+1||0,\widetilde{N-2},N|\\
&+2|\widetilde{N-2},N-1,N||0,\widetilde{N-2},N+1|=0.
\end{aligned} \tag{4.4.17}$$

$$\begin{aligned}
&D_tg\cdot f+D_x^3g\cdot f+3D_xD_yg\cdot f\\
&=g_tf-gf_t+g_{xxx}f-3g_{xx}f_x+3g_xf_{xx}-gf_{xxx}+3(g_{xy}f-g_xf_y-g_yf_x+f_{xy}g)\\
&=(g_t+g_{xxx}+3g_{xy})f+g(-f_t-f_{xxx}+3f_{xy})+3f_x(-g_{xx}-g_y)+3g_x(f_{xx}-f_y)\\
&=6[-|\widetilde{N-3},N-1,N,N+1||0,\widetilde{N-3},N-2,N-1|\\
&\quad-|\widetilde{N-3},N-2,N-1,N||0,\widetilde{N-3},N-1,N+1|\\
&\quad+|\widetilde{N-3},N-2,N-1,N+1||0,\widetilde{N-3},N-1,N|]\\
&\quad+6[|\widetilde{N-2},N,N+2||0,\widetilde{N-2},N-1|+|\widetilde{N-2},N-1,N||0,\widetilde{N-2},N+2|\\
&\quad-|\widetilde{N-2},N-1,N+2||0,\widetilde{N-2},N|]=0.
\end{aligned} \tag{4.4.18}$$

4.4.2 Wronskian method for the mKP equation with self-consistent sources

In this section, we will derive the solution in the Wronskian form for mKPESCS.

The Wronskian form solutions for the mKPESCS can be written as (4.4.3),(4.4.4) and

$$h_m=-\sqrt{2(k_m+q_m)\beta_m(t)}e^{\xi_m-\eta_m}\begin{vmatrix}
\psi_1 & \partial\psi_1 & \cdots & \partial^{N-2}\psi_1 & 0\\
\psi_2 & \partial\psi_2 & \cdots & \partial^{N-2}\psi_2 & 0\\
\cdots & \cdots & \cdots & \cdots & \cdots\\
\psi_{m-1} & \partial\psi_{m-1} & \cdots & \partial^{N-2}\psi_{m-1} & 0\\
\psi_m & \partial\psi_m & \cdots & \partial^{N-2}\psi_m & 1\\
\psi_{m+1} & \partial\psi_{m+1} & \cdots & \partial^{N-2}\psi_{m+1} & 0\\
\cdots & \cdots & \cdots & \cdots & \cdots\\
\psi_N & \partial\psi_N & \cdots & \partial^{N-2}\psi_N & 0
\end{vmatrix}, \tag{4.4.19}$$

$$s_m = \sqrt{2(k_m+q_m)\beta_m(t)} \begin{vmatrix} \partial\phi_1 & \partial^2\phi_1 & \cdots & \partial^{N-1}\phi_1 & 0 \\ \partial\phi_2 & \partial^2\phi_2 & \cdots & \partial^{N-1}\phi_2 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \partial\phi_{m-1} & \partial^2\phi_{m-1} & \cdots & \partial^{N-1}\phi_{m-1} & 0 \\ \partial\phi_m & \partial^2\phi_m & \cdots & \partial^{N-1}\phi_m & 1 \\ \partial\phi_{m+1} & \partial^2\phi_{m+1} & \cdots & \partial^{N-1}\phi_{m+1} & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \partial\phi_N & \partial^2\phi_N & \cdots & \partial^{N-1}\phi_N & 0 \end{vmatrix}, \tag{4.4.20}$$

where

$$\phi_j = e^{\xi_j} + (-1)^{j-1}e^{-\eta_j}, \tag{4.4.21}$$

$$\psi_j = (k_m - k_j)(k_j + q_m)e^{\xi_j} + (-1)^{j-1}(q_m - q_j)(q_j + k_m)e^{-\eta_j}, (j < m), \tag{4.4.22}$$

$$\psi_j = (k_j - k_m)(k_j + q_m)e^{\xi_j} + (-1)^{j-1}(q_j - q_m)(q_j + k_m)e^{-\eta_j}, (j > m). \tag{4.4.23}$$

First, we show that the Wronskian determinants f, g and h_m, s_m satisfy the bilinear equation (4.3.2) and (4.3.4). Expanding f, g and f_t, g_t by the mth row, we have

$$f = \sum_{j=1}^{N}(-1)^{m+j}\partial^{j-1}(e^{\xi_m} + (-1)^{m-1}e^{-\eta_m})A_{mj}, \tag{4.4.24}$$

$$f_t = \sum_{j=1}^{N}(-1)^{m+j}\partial^{j-1}[(-4k_j^3 - \beta_j(t))e^{\xi_m} + (-1)^{m-1}(4q_j^3 + \beta_j(t))e^{-\eta_m}]A_{mj}, \tag{4.4.25}$$

$$g = \sum_{l=1}^{N}(-1)^{m+l}\partial^{l}[e^{\xi_m} + (-1)^{m-1}e^{-\eta_m}]C_{ml}, \tag{4.4.26}$$

$$g_t = \sum_{l=1}^{N}(-1)^{m+l}\partial^{l}[(-4k_l^3 - \beta_l(t))e^{\xi_m} + (-1)^{m-1}(4q_l^3 + \beta_l(t))e^{-\eta_m}]C_{ml}, \tag{4.4.27}$$

where A_{mj} and C_{ml} are the cofactor of f and g respectively. Obviously $C_{mN} = A_{m1}$.

In section 3.1, We have shown that f, g with $\beta_j(t) = 0, (j = 1, 2, \cdots, N)$ satisfies the bilinear equation of mKP equation (4.4.1). So, what we should do is to prove the sum of all coefficient for a fixed $\beta_j(t)$ at the two side of (4.3.3) are equal.

Without loss of generality, the following discussion will be restricted the case of $\beta_m(t)$. Because there is only the first term $D_t g \cdot f = g_t f - g f_t$ including $\beta_m(t)$ and note the equality

$$[\partial^l(e^{\xi_m} - (-1)^{m-1}e^{-\eta_m})][\partial^{j-1}(e^{\xi_m} + (-1)^{m-1}e^{-\eta_m})]$$

$$- [\partial^l(e^{\xi_m} + (-1)^{m-1}e^{-\eta_m})][\partial^{j-1}(e^{\xi_m} - (-1)^{m-1}e^{-\eta_m})]$$

$$= 2(-1)^{m-1}e^{\xi_m-\eta_m}[k_m^l(-q_m)^{j-1} - k_m^{j-1}(-q_m)^l], \tag{4.4.28}$$

then the term for $\beta_m(t)$ at the left side of (4.3.3) can be written as

$$-2\beta_m(t)(-1)^{m-1}e^{\xi_m-\eta_m}\{\sum_{j=1}^{N-1}\sum_{l=1}^{N-1}(-1)^{l+j}[k_m^l(-q_m)^{j-1} - k_m^{j-1}(-q_m)^l]C_{ml}A_{mj}$$

$$+\sum_{l=1}^{N-1}(-1)^{l+N}[k_m^l(-q_m)^{N-1} - k_m^{N-1}(-q_m)^l]C_{ml}A_{mN}$$

$$+\sum_{j=1}^{N-1}(-1)^{j+N}[k_m^N(-q_m)^{j-1} - k_m^{j-1}(-q_m)^N]C_{mN}A_{mj}$$

$$+[k_m^N(-q_m)^{N-1} - k_m^{N-1}(-q_m)^N]C_{mN}A_{mN}\}. \tag{4.4.29}$$

By means of the general determinant identity

$$|Q,a,b||Q,c,d| - |Q,a,c||Q,b,d| + |Q,a,d||Q,b,c| = 0, \tag{4.4.30}$$

where Q is an $(N-1)\times(N-3)$ matrix and a, b, c and d represent $N-1$ column vectors, it is not difficult to prove that

$$C_{mj} = |M(j), N|, \quad j = 1, 2, \cdots, N-1, \tag{4.4.31}$$

$$A_{mj} = |0, M(j)|, \quad j = 1, 2, \cdots, N-1, \tag{4.4.32}$$

$$C_{mj}A_{m,l+1} - C_{ml}A_{m,j+1} = |0, M(l,j), N|C_{mN}, \quad (1 \le l < j \le N-3), \tag{4.4.33}$$

$$C_{m,N-1}A_{m,j+1} - C_{mj}A_{mN} = |0, M(j, N-1), N|C_{mN}, \quad j = 1, 2, \cdots, N-2, \tag{4.4.34}$$

where the matrix $M(l, j)$ is defined by

$$M(l,j) = |1, 2, \cdots, l-1, l+1, \cdots, j-1, j+1, \cdots, N-1|_{(N-1)\times(N-3)}, \tag{4.4.35}$$

$$M(j) = |1, 2, \cdots, j-1, j+1, \cdots, N-1|_{(N-1)\times(N-2)}. \tag{4.4.36}$$

Using (4.4.31)-(4.4.36), the expression (4.4.29) becomes

$$-2\beta_m(t)(-1)^{m-1}e^{\xi_m-\eta_m}\{\sum_{l=1}^{N-3}\sum_{j=l+1}^{N-2}(-k_mq_m)^l[q_m^{j-l} - (-k_m)^{j-l}]|0, M(l,j), N|$$

$$+\sum_{j=1}^{N-2}(-k_mq_m)^j[q_m^{N-1-j} - (-k_m)^{N-1-j}]|0, M(j, N-1), N|$$

$$+\sum_{l=1}^{N-1}[q_m^l-(-k_m)^l]C_{ml}$$

$$+\sum_{j=1}^{N-1}(-k_mq_m)^{j-1}[q_m^{N+1-j}-(-k_m)^{N+1-j}]A_{mj}$$

$$+(-k_mq_m)^{N-1}(k_m+q_m)A_{mN}\}C_{mN}. \tag{4.4.37}$$

Now we turn about h_m and s_m. Obviously, from (4.4.20) we have

$$s_m=\sqrt{2(k_m+q_m)\beta_m(t)}(-1)^{m+N}C_{mN}. \tag{4.4.38}$$

While from (4.4.19), h_m can be written as

$$h_m=-\sqrt{2(k_m+q_m)\beta_m(t)}e^{\xi_m-\eta_m}\tilde{h}_m, \tag{4.4.39}$$

where $\tilde{h}_m$ is a $N\times N$ determinant

$$\tilde{h}_m=\begin{vmatrix} -L\phi_1 & -L\phi_1^{(1)} & \cdots & -L\phi_1^{(N-2)} & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ -L\phi_{m-1} & -L\phi_m^{(1)} & \cdots & -L\phi_m^{(N-2)} & 0 \\ L\phi_m & L\phi_m^{(1)} & \cdots & L\phi_m^{(N-2)} & 1 \\ L\phi_{m+1} & L\phi_{m+1}^{(1)} & \cdots & L\phi_{m+1}^{(N-2)} & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ L\phi_N & L\phi_N^{(1)} & \cdots & L\phi_N^{(N-2)} & 0 \end{vmatrix}$$

$$=(-1)^{m-1}|L\cdot 0,L\cdot 1,L\cdot 2,\cdots,L\cdot(N-2),\tau_m|$$

$$=|b\cdot 0+a\cdot 1+2,b\cdot 1+a\cdot 2+3,b\cdot 2+a\cdot 3+4,\cdots,b\cdot(N-2)+a\cdot(N-1)+N,\tau_m|$$

$$L=b+a\partial+\partial^2,\ \ b=-k_mq_m,\ \ a=q_m-k_m,\ \ \tau_m=(\delta_{m,1},\delta_{m,2},\cdots,\delta_{m,N})^T. \tag{4.4.40}$$

By simple analysis for $N\times N$ determinant

$$|F(l,j),\tau_m|=|0,1,\cdots,l-1,l+1,\cdots,j-1,j+1,j+2,\cdots,N,\tau_m|, \tag{4.4.41}$$

we obtain

$$\tilde{h}_m=(-1)^{m-1}\sum_{l=0}^{N-1}\sum_{j=l+1}^{N}|F(l,j),\tau_m|(-k_mq_m)^l$$

$$[(q_m-k_m)^{j-l-1}-C_{j-l-2}^1(q_m-k_m)^{j-l-3}(-k_mq_m)+C_{j-l-3}^2(q_m-k_m)^{j-l-5}(-k_mq_m)^2$$

$$-C^3_{j-l-4}(q_m-k_m)^{j-l-7}(-k_mq_m)^3+\cdots$$

$$+\begin{cases}(-1)^{\frac{j-l-1}{2}}(-k_mq_m)^{\frac{j-l-1}{2}}] & \text{if } j-l \text{ is odd}\\ (-1)^{\frac{j-l-2}{2}}C^{\frac{j-l-2}{2}}_{\frac{j-l}{2}}(-k_mq_m)^{\frac{j-l-2}{2}}(q_m-k_m)] & \text{if } j-l \text{ is even}\end{cases} \tag{4.4.42}$$

We show further the algebraic sum after $|F(l,j),\tau_m|$ be expressed as

$$(-k_mq_m)^l\frac{q_m^{j-l}-(-k_m)^{j-l}}{q_m+k_m}. \tag{4.4.43}$$

Because of comparing the coefficients of the terms $(-k_mq_m)^lq_m^{j-l-n-1}(-k_m)^n$ in (4.4.42) and (4.4.43), we have

$$C^n_{j-l-1}-C^1_{j-l-2}C^{n-1}_{j-l-3}+C^2_{j-l-3}C^{n-2}_{j-l-5}+\cdots+(-1)^nC^n_{j-l-n-1}=1,\ \ n\le[\frac{1}{2}(j-l-1)], \tag{4.4.44}$$

or

$$\sum_{k=0}^{n}(-1)^kC^k_nC^n_{j-l-k-1}=1. \tag{4.4.45}$$

But

$$\sum_{k=0}^{n}(-1)^kC^k_nC^n_{j-l-k-1}=C^n_{j-l-1}+\sum_{k=1}^{n-1}(-1)^k(C^k_{n-1}+C^{k-1}_{n-1})C^n_{j-l-k-1}+(-1)^nC^n_{j-l-n-1}$$

$$=\sum_{k=0}^{n-1}(-1)^kC^k_{n-1}(C^n_{j-l-k-1}-C^n_{j-l-k-2})$$

$$=\sum_{k=0}^{n-1}(-1)^kC^k_{n-1}C^{n-1}_{j-l-k-2}. \tag{4.4.46}$$

Using the induction for n of expression (4.4.46), it may be seen that the equality (4.4.44) or (4.4.45) is true. As a result, we obtain immediately

$$h_m=(-1)^m\sqrt{2(k_m+q_m)\beta_m(t)}e^{\xi_m-\eta_m}$$

$$\{\sum_{l=0}^{N-3}\sum_{j=l+1}^{N-2}(-k_mq_m)^l\left(\frac{q_m^{j-l}-(-k_m)^{j-l}}{q_m+k_m}\right)|P(l,j),N-1,N,\tau_m|$$

$$+\sum_{l=0}^{N-2}(-k_mq_m)^l\left(\frac{q_m^{N-1-l}-(-k_m)^{N-1-l}}{q_m+k_m}\right)|P(l),N,\tau_m|$$

$$+\sum_{l=0}^{N-2}(-k_mq_m)^l\left(\frac{q_m^{N-l}-(-k_m)^{N-l}}{q_m+k_m}\right)|P(l),N-1,\tau_m|$$

$$+(-k_mq_m)^{N-1}|\widehat{N-2},\tau_m|\}$$

$$= (-1)^m \sqrt{2(k_m + q_m)\beta_m(t)} e^{\xi_m - \eta_m}$$

$$\{\sum_{l=1}^{N-3} \sum_{j=l+1}^{N-2} (-k_m q_m)^l \left(\frac{q_m^{j-l} - (-k_m)^{j-l}}{q_m + k_m} \right) |P(l,j), N-1, N, \tau_m|$$

$$+ \sum_{j=1}^{N-1} \left(\frac{q_m^j - (-k_m)^j}{q_m + k_m} \right) |P(0,j), N-1, N, \tau_m|$$

$$+ \sum_{l=1}^{N-2} (-k_m q_m)^l \left(\frac{q_m^{N-1-l} - (-k_m)^{N-1-l}}{q_m + k_m} \right) |P(l), N, \tau_m|$$

$$+ \sum_{l=0}^{N-2} (-k_m q_m)^l \left(\frac{q_m^{N-l} - (-k_m)^{N-l}}{q_m + k_m} \right) |P(l), N-1, \tau_m|$$

$$+ (-k_m q_m)^{N-1} |\widehat{N-2}, \tau_m|\} \tag{4.4.47}$$

here $P(l,j)$ is the $N \times (N-3)$ matrix without l column and j column, $P(l)$ is the $N \times N-2$ matrix without l column. Obviously we have

$$|P(l,j), N-1, N, \tau_m| = |0, M(l,j), N|(-1)^{m+N}, |P(l), N, \tau_m| = |0, M(l, N-1), N|(-1)^{m+N},$$

$$|P(l), N-1, \tau_m| = |0, M(l)|(-1)^{m+N}, \quad P|(0,j), N-1, N, \tau_m| = C_{mj}(-1)^{m+N},$$

$$|N-2, \tau_m| = A_{mN}(-1)^{m+N}. \tag{4.4.48}$$

So

$$h_m s_m = (-1)^m 2\beta_m(t) e^{\xi_m - \eta_m}$$

$$\{\sum_{l=1}^{N-3} \sum_{j=l+1}^{N-2} (-k_m q_m)^l [q_m^{j-l} - (-k_m)^{j-l}] |0, M(l,j), N|$$

$$+ \sum_{j=1}^{N-1} [q_m^j - (-k_m)^j] C_{mj}$$

$$+ \sum_{l=1}^{N-2} (-k_m q_m)^l [q_m^{N-1-l} - (-k_m)^{N-1-l}] |0, M(l, N-1), N|$$

$$+ \sum_{l=0}^{N-2} (-k_m q_m)^l [q_m^{N-l} - (-k_m)^{N-l}] A_{ml}$$

$$+ (-k_m q_m)^{N-1} (k_m + q_m) A_{mN}\} C_{mN} \tag{4.4.49}$$

wherefore the term for $\beta_m(t)$ at the right side of (4.3.3) equal (4.4.37). That is to say that Wronskian form (4.4.3), (4.4.4) and (4.4.19),(4.4.20) satisfy equation (4.3.3).

Next, we will prove h_m and f satisfy (4.3.4). Using the abbreviated notation, it can be obtained

$$h_{m,y} = (-1)^m \sqrt{2(k_m + q_m)\beta_m(t)} e^{\xi_m - \eta_m} [-|L(\widehat{N-4}), L(N-2), L(N-1), \tau_m| + |L(\widehat{N-3}), LN, \tau_m| + (k_m^2 + q_m^2)|L(\widehat{N-2}), \tau_m|] \tag{4.4.50}$$

$$h_{m,x} = (-1)^m \sqrt{2(k_m + q_m)\beta_m(t)} e^{\xi_m - \eta_m} [|L(\widehat{N-3}), L(N-1), \tau_m| + (k_m - q_m)|L(\widehat{N-2}), \tau_m|], \tag{4.4.51}$$

$$h_{m,xx} = (-1)^m \sqrt{2(k_m + q_m)\beta_m(t)} e^{\xi_m - \eta_m} [|L(\widehat{N-4}), L(N-2), L(N-1), \tau_m| + |L(\widehat{N-3}), LN, \tau_m| + 2(k_m - q_m)|L(\widehat{N-3}), L(N-1), \tau_m| + (k_m - q_m)^2 |L(\widehat{N-2}), \tau_m|]. \tag{4.4.52}$$

Substituting (4.4.50)-(4.4.52) and Wronskian f and its related derivatives (4.4.7) into bilinear equation (4.3.4) gives

$$[-|L(\widehat{N-4}), L(N-2), L(N-1), \tau_m| + (q_m - k_m)|L(\widehat{N-3}), L(N-1), \tau_m| + k_m q_m |L(\widehat{N-2}), \tau_m|]|\widehat{N-1}| + [|L(\widehat{N-3}), L(N-1), \tau_m| - (q_m - k_m)|L(\widehat{N-2}), \tau_m|]|\widehat{N-2}, N| - |L(\widehat{N-2}), \tau_m||\widehat{N-2}, N+1| = 0, \tag{4.4.53}$$

But we can work out that

$$\begin{aligned}
&(q_m + k_m)|L(\widehat{N-4}), L(N-2), L(N-1), \tau_m| \\
&= \sum_{l=0}^{N-3} \sum_{j=l+1}^{N-2} (-k_m q_m)^l [q_m^{j-l} - (-k_m)^{j-l}][(q_m - k_m)^2 + k_m q_m]|P(l,j), N-1, N, \tau_m| \\
&+ \sum_{l=0}^{N-3} \sum_{j=l+1}^{N-2} (-k_m q_m)^l [q_m^{j-l} - (-k_m)^{j-l}](q_m - k_m)|P(l,j), N-1, N+1, \tau_m| \\
&+ \sum_{l=0}^{N-3} \sum_{j=l+1}^{N-2} (-k_m q_m)^l [q_m^{j-l} - (-k_m)^{j-l}]|P(l,j), N, N+1, \tau_m| \\
&+ \sum_{l=0}^{N-2} (-k_m q_m)^{l+2} [q_m^{N-2-l} - (-k_m)^{N-2-l}]|P(l), N-1, \tau_m| \\
&+ \sum_{l=0}^{N-2} (-k_m q_m)^{l+1} [q_m^{N-2-l} - (-k_m)^{N-2-l}](q_m - k_m)|P(l), N, \tau_m| \\
&+ \sum_{l=0}^{N-2} (-k_m q_m)^{l+1} [q_m^{N-2-l} - (-k_m)^{N-2-l}]|P(l), N+1, \tau_m|,
\end{aligned} \tag{4.4.54}$$

$$(q_m + k_m)|L(\widehat{N-3}), L(N-1), \tau_m|$$

$$= \sum_{l=0}^{N-3}\sum_{j=l+1}^{N-2} (-k_m q_m)^l [q_m^{j-l} - (-k_m)^{j-l}](q_m - k_m)|P(l,j), N-1, N, \tau_m|$$

$$+ \sum_{l=0}^{N-3}\sum_{j=l+1}^{N-2} (-k_m q_m)^l [q_m^{j-l} - (-k_m)^{j-l}]|P(l,j), N-1, N+1, \tau_m|$$

$$+ \sum_{l=0}^{N-2} (-k_m q_m)^{l+1} [q_m^{N-1-l} - (-k_m)^{N-1-l}]|P(l), N-1, \tau_m|$$

$$+ \sum_{l=0}^{N-2} (-k_m q_m)^{l} [q_m^{N-1-l} - (-k_m)^{N-1-l}](q_m - k_m)|P(l), N, \tau_m|$$

$$+ \sum_{l=0}^{N-2} (-k_m q_m)^{l} [q_m^{N-1-l} - (-k_m)^{N-1-l}]|P(l), N+1, \tau_m|. \qquad (4.4.55)$$

Inserting (4.4.47), (4.4.54) and (4.4.55) into the left side of (4.4.53) leaves only the terms

$$\sum_{l=0}^{N-3}\sum_{j=l+1}^{N-2} (-k_m q_m)^l [q_m^{j-l} - (-k_m)^{j-l}](-|P(l,j), N-1, N, \tau_m||\widehat{N-2}, N+1|$$

$$+ |P(l,j), N-1, N+1, \tau_m||\widehat{N-2}, N| - |P(l,j), N, N+1, \tau_m||\widehat{N-1}|)$$

$$+ \sum_{l=0}^{N-2} (-k_m q_m)^l [q_m^{N+1-l} - (-k_m)^{N+1-l}](|P(l), N, \tau_m||\widehat{N-1}| - |P(l), N-1, \tau_m||\widehat{N-2}, N|)$$

$$+ \sum_{l=0}^{N-2} (-k_m q_m)^l [q_m^{N-l} - (-k_m)^{N-l}](|P(l), N+1, \tau_m||\widehat{N-1}| - |P(l), N-1, \tau_m||\widehat{N-2}, N+1|)$$

$$+ \sum_{l=0}^{N-2} (-k_m q_m)^l [q_m^{N-1-l} - (-k_m)^{N-1-l}](|P(j), N+1, \tau_m||\widehat{N-2}, N| - |P(m), N, \tau_m||\widehat{N-2}, N+1|)$$

$$+ (-k_m q_m)^{N-1}(q_m + k_m)|\widehat{N-2}, \tau_m|[k_m q_m|\widehat{N-1}| - (q_m - k_m)|\widehat{N-2}, N| - |\widehat{N-2}, N+1|], \qquad (4.4.56)$$

noting that

$$-|P(l,j), N-1, N, \tau_m||\widehat{N-2}, N+1| + |P(l,j), N-1, N+1, \tau_m||\widehat{N-2}, N|$$

$$-|P(l,j), N, N+1, \tau_m||\widehat{N-1}|$$

$$= (-1)^{N+1}\begin{vmatrix} P(l,j) & O & 0 & 0 & N-1 & N & N+1 & \tau_m \\ O & P(l,j) & l & j & N-1 & N & N+1 & \tau_m \end{vmatrix}$$

$$- |P(l,j), N-1, N, N+1||\widehat{N-2}, \tau_m|$$

$$= -|P(l,j), N-1, N, N+1||\widehat{N-2}, \tau_m|, \tag{4.4.57}$$

$$|P(l), N, \tau_m||\widehat{N-1}| - |P(l), N-1, \tau_m||\widehat{N-2}, N|$$

$$= \begin{vmatrix} P(l) & O & 0 & N-1 & N & \tau_m \\ O & P(l) & l & N-1 & N & \tau_m \end{vmatrix} - |P(l), N-1, N||\widehat{N-2}, \tau_m|$$

$$= -|P(l), N-1, N||\widehat{N-2}, \tau_m|, \tag{4.4.58}$$

$$[|P(l), N+1, \tau_m||\widehat{N-1}| - |P(l), N-1, \tau_m||\widehat{N-2}, N+1|]$$

$$= \begin{vmatrix} P(l) & O & 0 & N-1 & N+1 & \tau_m \\ O & P(l) & l & N-1 & N+1 & \tau_m \end{vmatrix} - |P(l), N-1, N+1||\widehat{N-2}, \tau_m|$$

$$= -|P(l), N-1, N+1||\widehat{N-2}, \tau_m|, \tag{4.4.59}$$

$$[|P(l), N+1, \tau_m||\widehat{N-1}| - |P(l), N-1, \tau_m||\widehat{N-2}, N+1|]$$

$$= \begin{vmatrix} P(l) & O & 0 & N & N+1 & \tau_m \\ O & P(l) & l & N & N+1 & \tau_m \end{vmatrix} - |P(l), N, N+1||\widehat{N-2}, \tau_m|$$

$$= -|P(l), N, N+1||\widehat{N-2}, \tau_m|, \tag{4.4.60}$$

then (4.4.56) reduce to

$$\begin{aligned}(-|\widehat{N-2}, \tau_m|)\{&\sum_{l=0}^{N-3}\sum_{j=l+1}^{N-2}(-k_m q_m)^l[q_m^{j-l} - (-k_{m1})^{j-l}]|P(l,j), N-1, N, N+1| \\ &+ \sum_{l=0}^{N-2}(-k_m q_m)^l[q_m^{N+1-l} - (-k_m)^{N+1-l}]|P(l), N-1, N| \\ &+ \sum_{l=0}^{N-2}(-k_m q_m)^l[q_m^{N-l} - (-k_m)^{N-l}]|P(l), N-1, N+1| \\ &+ \sum_{l=0}^{N-2}(-k_m q_m)^l[q_m^{N-1-l} - (-k_m)^{N-1-l}]|P(l), N, N+1| \\ -(-k_m q_m)^{N-1}&(q_m + k_m)[k_m q_m|\widehat{N-1}| + (k_m - q_m)|\widehat{N-2}, N| - |\widehat{N-2}, N+1|]\},\end{aligned} \tag{4.4.61}$$

which is just $-|\widehat{N-2}, \tau_m||L(\widehat{N+1})| = 0$.

At last, the remained problem is to prove that Wronskian s_m and g satisfy equation (4.3.5).

$$s_{m,x} = \sqrt{2(k_m + q_m)\beta_m(t)}|\widetilde{N-2}, N, \tau_m|,$$

$$s_{m,xx} = \sqrt{2(k_m + q_m)\beta_m(t)}[|\widetilde{N-3}, N-1, N, \tau_m| + |\widetilde{N-2}, N+1, \tau_m|], \tag{4.4.62}$$

$$s_{m,y} = \sqrt{2(k_m+q_m)\beta_m(t)}[-|\widetilde{N-3}, N-1, N, \tau_m| + |\widetilde{N-2}, N+1, \tau_m|]. \tag{4.4.63}$$

Substituting (4.4.12)-(4.4.14) and (4.4.62)-(4.4.63) into (4.3.5)

$$D_y s_m \cdot g + D_x^2 s_m \cdot g = s_{m,y}g - s_m g_y + s_{m,xx}g - 2s_{m,x}g_x + s_m g_{xx}$$

$$= (s_{m,y} + s_{m,xx})g - 2s_{m,x}g_x + s_m(g_{xx} - g_y)$$

$$= \sqrt{2(k_m+q_m)\beta_m(t)}[|\widetilde{N-2}, N+1, \tau_m||\widetilde{N-2}, N-1, N|$$

$$|\widetilde{N-2}, N, \tau_m||\widetilde{N-2}, N-1, N+1| + |\widetilde{N-2}, N-1, \tau_m||\widetilde{N-2}, N, N+1|] = 0. \tag{4.4.64}$$

4.5 Coincidence of the solutions

By now, we have found two kinds solutions of the bilinear equations (4.3.1)-(4.3.5), where (4.4.3)-(4.4.4) and (4.4.19)-(4.4.20) are just verified whereas (4.3.54)-(4.3.57) is only conjectured. In this section, we will show these two kinds of solutions are the same for recovering the N-soliton solutions from the transformation (4.3.1).

By virtue of the addition rule of determinants, (4.4.3) can be represented by the sum of 2^{N-1} Vandermonde determinants. So we have

$$f = \sum_{\epsilon=0,1}(2\epsilon_2-1)(2\epsilon_4-1)\cdots(2\epsilon_{[\frac{N}{2}]}-1)\Delta(\epsilon_1k_1+(\epsilon_1-1)q_1, \epsilon_2k_2+(\epsilon_2-1)q_2, \cdots, \epsilon_Nk_N+(\epsilon_N-1)q_N)$$

$$\exp\{\sum_{j=1}^{N}[\epsilon_j\xi_j + (\epsilon_j-1)\eta_j]\}$$

$$= \sum_{\epsilon=0,1}\prod_{k=1}^{[\frac{N}{2}]}(2\epsilon_{2k}-1)\prod_{1\le j<l}^{N}[\epsilon_lk_l + (\epsilon_l-1)q_l - \epsilon_jk_j - (\epsilon_j-1)q_j]\exp\{\sum_{j=1}^{N}[\epsilon_j\xi_j + (\epsilon_j-1)\eta_j]\}$$

$$= (-1)^{\frac{N(N-1)}{2}}\sum_{\epsilon=0,1}\prod_{1\le j<l}(2\epsilon_l-1)[\epsilon_jk_j + (\epsilon_j-1)q_j - \epsilon_lk_l - (\epsilon_l-1)q_l]\exp\{\sum_{j=1}^{N}[\epsilon_j\xi_j + (\epsilon_j-1)\eta_j]\}. \tag{4.5.1}$$

where $\Delta(\epsilon_1k_1+(\epsilon_1-1)q_1, \epsilon_2k_2+(\epsilon_2-1)q_2, \cdots, \epsilon_Nk_N+(\epsilon_N-1)q_N)$ denotes an $N\times N$ Vandermonde determinant with the entries $\epsilon_1k_1+(\epsilon_1-1)q_1, \epsilon_2k_2+(\epsilon_2-1)q_2, \cdots, \epsilon_Nk_N+(\epsilon_N-1)q_N$ and the sum over $\epsilon = 0,1$ refers to each of the $\epsilon_j = 0,1, (j = 1,2,\cdots,N)$.

Noticing that

$$\frac{(2\epsilon_l-1)[\epsilon_jk_j + (\epsilon_j-1)q_j - \epsilon_lk_l - (\epsilon_l-1)q_l]}{q_j - q_l}$$

$$= \left(\frac{k_j+q_l}{q_l-q_j}\right)^{(1-\epsilon_l)\epsilon_j} \left(\frac{k_l+q_j}{q_l-q_j}\right)^{(1-\epsilon_j)\epsilon_l} \left(\frac{k_l-k_j}{q_l-q_j}\right)^{\epsilon_j\epsilon_l}$$

$$= \left(\frac{k_j+q_l}{q_l-q_j}\right)^{\epsilon_j} \left(\frac{k_l+q_j}{q_l-q_j}\right)^{\epsilon_l} \left[\frac{(k_l-k_j)(q_l-q_j)}{(k_j+q_l)(k_l+q_j)}\right]^{\epsilon_j\epsilon_l}, \tag{4.5.2}$$

and

$$\prod_{1\le j<l\le N} \left[\frac{(k_l-k_j)(q_l-q_j)}{(k_j+q_l)(k_l+q_j)}\right]^{\epsilon_j\epsilon_l} = \exp\left(\sum_{1\le j<l}^{N} \epsilon_j\epsilon_l A_{jl}\right),$$

(4.5.1) becomes

$$f = \prod_{1\le j<l}^{N} (q_l-q_j) e^{\sum_{j=1}^{N}(-\eta_j)} \sum_{\epsilon=0,1} \exp\left[\sum_{j=1}^{N} \epsilon_j(\xi_j'+\eta_j') + \sum_{1\le j<l}^{N} \epsilon_j\epsilon_l A_{jl}\right], \tag{4.5.3}$$

where

$$e^{\xi_j} \prod_{j\ne l}(k_j+q_l) = e^{\xi_j'}, \quad e^{\eta_j} \prod_{j>l}(q_j-q_l)^{-1} \prod_{l>j}(q_l-q_j)^{-1} = e^{-\eta_j'}. \tag{4.5.4}$$

Similar to g

$$g = \prod_{1\le j<l}^{N} (q_l-q_j) e^{\sum_{j=1}^{N}(-\eta_j-b_j)} \sum_{\epsilon=0,1} \exp\left[\sum_{j=1}^{N} \epsilon_j(\xi_j'+a_j+\eta_j'+b_j) + \sum_{1\le j<l}^{N} \epsilon_j\epsilon_l A_{jl}\right]. \tag{4.5.5}$$

It may be seen that the soliton solution constructed from (4.5.3) and (4.5.5) is the same as the one from (4.3.54)-(4.3.57) with $e^{\alpha_j} = -\frac{k_j}{q_j}$, $e^{\gamma_j} = 1$.

We can deal with s_m and h_m in a similar way to f, g. (4.4.20) can be rewritten as

$$s_m = \sqrt{2(k_m+q_m)\beta_m(t)}$$

$$\cdot \begin{vmatrix} \partial\phi_1 & \partial^2\phi_1 & \cdots & \partial^{N-1}\phi_1 & 0 \\ \partial\phi_2 & \partial^2\phi_2 & \cdots & \partial^{N-1}\phi_2 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \partial\phi_{m-1} & \partial^2\phi_{m-1} & \cdots & \partial^{N-1}\phi_{m-1} & 0 \\ \partial\phi_m & \partial^2\phi_m & \cdots & \partial^{N-1}\phi_m & 1 \\ \partial\phi_{m+1}' & \partial^2\phi_{m+1}' & \cdots & \partial^{N-1}\phi_{m+1}' & \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \partial\phi_N' & \partial^2\phi_N' & \cdots & \partial^{N-1}\phi_N' & 0 \end{vmatrix}, \tag{4.5.6}$$

where $\phi_j' = (-i)[e^{\xi_j+\frac{\pi}{2}i} + (-1)^j e^{-\eta_j-\frac{\pi}{2}i}]$. So

$$s_m = \sqrt{2(k_m+q_m)\beta_m(t)} \prod_{1\le j<l, j,l\ne m} (q_j-q_l) \exp\left[\sum_{j=1,j\ne m}^{N} (-\eta_j-b_j)\right]$$

$$\sum_{\epsilon=0,1}\exp[\sum_{j<m}\epsilon_j(\xi_j'+\eta_j'+C_{mj})+\sum_{j>m}\epsilon_j(\xi_j'+a_j+\eta_j'+b_j+i\pi+C_{jm})+\sum_{1\le j<l}^{N}\epsilon_j\epsilon_l A_{jl}]. \quad (4.5.7)$$

From (4.4.19), we have

$$h_m=-\sqrt{2(k_m+q_m)\beta_m(t)}e^{\xi_m-\eta_m}$$

$$\begin{vmatrix} \psi_1 & \partial\psi_1 & \cdots & \partial^{N-2}\psi_1 & 0 \\ \psi_2 & \partial\psi_2 & \cdots & \partial^{N-2}\psi_2 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \psi_{m-1} & \partial\psi_{m-1} & \cdots & \partial^{N-2}\psi_{m-1} & 0 \\ \psi_m & \partial\psi_m & \cdots & \partial^{N-2}\psi_m & 1 \\ \psi_{m+1}' & \partial\psi_{m+1}' & \cdots & \partial^{N-2}\psi_{m+1}' & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \psi_N' & \partial\psi_N' & \cdots & \partial^{N-2}\psi_N' & 0 \end{vmatrix}, \quad (4.5.8)$$

where

$$\psi_j'=(-i)[(k_j-k_m)(k_j+q_m)e^{\xi_j+\frac{\pi}{2}i}+(-1)^j(q_j-q_m)(q_j+k_m)e^{-\eta_j-\frac{\pi}{2}i}],(j>m). \quad (4.5.9)$$

It is easy to derive

$$h_m=-\sqrt{2(k_m+q_m)\beta_m(t)}(-1)^{N-m}(-i)^{N-m}\sum_{\epsilon=0,1}\prod_{k=1}^{[\frac{N}{2}]}(2\epsilon_{2k}-1)\prod_{1\le j<l}^{N}[\epsilon_l k_l+(\epsilon_l-1)q_l-\epsilon_j k_j-(\epsilon_j-1)q_j]$$

$$\exp\{\sum_{j=1}^{N}[\epsilon_j\xi_j+(\epsilon_j-1)\eta_j]\}\prod_{j<m}\{[(k_m-k_j)(k_j+q_m)]^{\epsilon_j}[(q_m-q_j)(q_j+k_m)]^{1-\epsilon_j}\}$$

$$\prod_{j>m}\{[(k_j-k_m)(k_j+q_m)]^{\epsilon_j}[(q_j-q_m)(q_j+k_m)]^{1-\epsilon_j}\}$$

$$=-\sqrt{2(k_m+q_m)\beta_m(t)}(-1)^{\frac{N(N-1)}{2}}\sum_{\epsilon=0,1}\prod_{1\le j<l}(2\epsilon_l-1)[\epsilon_j k_j+(\epsilon_j-1)q_j-\epsilon_l k_l-(\epsilon_l-1)q_l]$$

$$\exp[\sum_{j=1,j\ne m}^{N}(-\eta_j)]\prod_{j=1,j\ne m}\left[\frac{(k_j-k_m)(k_j+q_m)}{(q_j-q_m)(q_j+k_m)}\right]^{\epsilon_j}\prod_{j>m}[(q_j-q_m)(q_j+k_m)]$$

$$\prod_{j<m}[(q_m-q_j)(q_j+k_m)]\exp\{\sum_{j=1}^{N}[\epsilon_j\xi_j+(\epsilon_j-1)\eta_j]\}$$

$$=-\sqrt{2(k_m+q_m)\beta_m}\prod_{1\le j<l}(q_l-q_j)\prod_{j\ne m}(q_j+k_m)e^{\xi_m}\exp[\sum_{j=1}^{N}(-\eta_j)]$$

$$\sum_{\epsilon=0,1}\exp[\sum_{j<m}\epsilon_j(\xi_j^{'}+\eta_j^{'}+B_{mj})+\sum_{j>m}\epsilon_j(\xi_j^{'}+\eta_j^{'}+i\pi+B_{jm})+\sum_{1\le j<l}^{N}\epsilon_j\epsilon_l A_{jl}]. \tag{4.5.10}$$

From (4.5.3,4.5.5) and (4.5.7,4.5.10), we find that the solution Φ_j, Ψ_j gotten by Wronskian form is identical with the one gotten by Hirota method with $e^{\alpha_j}=-\frac{k_j}{q_j}$, $e^{\gamma_j}=1$.

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