文档：
中文分八章研究

Clifford 分析中高阶奇异积分和边值问题

摘 要

K.T.Vahlen[1] 在本世纪初创建了 Clifford 代数，它是一个可结合但不可交换的代数. Clifford 分析是上世纪七十年代新兴起的一个数学分支，研究的是从实向量空间映射到不可交换的实 Clifford 代数的函数理论，它在数学其它分支、场论、量子力学等方面有着重要的应用. 许多数学家对它给予很大关注. Robert P.Gilbert[2],F.Brack,R.Relanghe,F.Sommen[3], 闻国椿 [4], Lehuang Song[5], 黄沙等 [6-12] 将 Clifford 分析与单、多元复分析函数理论相对照，做了很多工作. 到上世纪末， Clifford 分析有了长足的发展.

本文在 Clifford 分析中考虑高阶奇异积分，函数的性质以及边值问题，全文共分八章.

第一到第三章讨论 Clifford 分析中高阶奇异积分. 在实分析中积分理论是非常重要的，它不但是现代数学的理论基础，而且是数学应用的主要工具之一. 但是有些实际问题只用正常积分是不够的，需要考虑奇异积分，甚至是高阶奇异积分. Hadamard[13] 在实分析中是用积分有限部分的思想，定义高阶奇异积分的. 例如考虑高阶奇异积分 $\int_a^b \frac{f(x)dx}{(b-x)^{k+\frac{1}{2}}}$, 它的有限部分如下定义

$$F.P.\int_a^b \frac{f(x)dx}{(b-x)^{k+\frac{1}{2}}} = \lim_{\varepsilon\to 0}\left[\int_a^{b-\varepsilon} \frac{f(x)dx}{(b-x)^{k+\frac{1}{2}}} - H(b,k,\varepsilon)\right],$$

其中

$$H(b,k,\varepsilon)=\sum_{j=0}^{k-1}\frac{f^{(j)}(b)}{j\,!}(x-b)^j.$$

主要思想是舍去不可积的部分，用其余部分定义高阶奇异积分. 在文 [14] 中，路见可直截了当地定义闭曲线 L 上的高阶奇异积分为 $\int_L\frac{f(t)}{(t-t_0)^2}dt=\int_L\frac{f'(t)}{t-t_0}dt$, 即定义 $\int_L(\frac{-1}{t-t_0})'f(t)dt$ $=\int_L(\frac{1}{t-t_0})f'(t)dt$, 这里 $(\frac{1}{t-t_0})$ 比 $(\frac{-1}{t-t_0})'$ 具有较低阶的奇性. 路先生关于高阶奇异积分的理论得到了数学理论界的广泛认同并在实际工作中得到了应用. 这是按归纳法思想用正常积分或弱奇异积分定义奇异积分，用低阶奇异积分来定义高阶奇异积分. 对高阶奇异积分的研究还有一些成果见文献 [14][15][16].

黄沙在文 [17] 中讨论了 Clifford 分析中 3 种高阶奇异积分，本文在文 [17] 的基础上，借助于高阶奇异积分的 Hadamard 主值的思想以及归纳法的思想，在第一到第三章讨论了几类拟 Bochner-Martinelli 型 (以下简称为拟 B-M 型) 高阶奇异积分的归纳定义、计算、性质及 Poincaré-Bertrand 置换公式等. 我们知道积分算子、积分方程在解边值问题及微分方程中都有重要应用. 所以这部分内容无论在数学理论上还是应用方面都有一定的意义.

第四章讨论了第二类积分方程的可解性和解的表示式. 积分方程问题是实际中非常重要的一个问题， Clifford 分析中积分方程的研究还是空白. M.Spivark 在他的专著 <<Calculus on Manifold>> 中，对 n 维实空间中流形上的函数，借助于单位分解的方法给出了广义积分的定义 (见文 [18])，这个定义是一维实广义积分定义的推广，它允许在流形上出现多个奇点. 我们是在 Clifford 分析中用类似的方法定义了广义积分. Robert P.Gilbert 在文献 [2] 中引入了交换因子的概念，部分的解决了 Clifford 分析中乘积不可交换的问题. 借助于这一思想我们设计了带交换因子的积分算子核，并研究第二类带交换因子核的积分方程的可解性和解的级数表达式.

第五和第六两章考虑了两类函数的边值问题. 在 Clifford 分析中的正则函数相当于单元复分析中所讲的全纯函数，它的定义首先是由文 [22] 给出的. 关于正则函数的 Cauchy 型奇异

积分及 Plemelj 公式已有研究，文 [20] 就利用上面的结果讨论了正则函数的一种特殊的边值问题. 文 [19] 定义了 Clifford 分析中的双正则函数和广义双正则函数，双正则函数和广义双正则函数是比正则函数更广泛的函数类，它们的研究是近年来函数论领域内的一个热门分支，受 [21] 的启发，本文第五和第六两章考虑了这两类函数的带位移的边值问题. 设计积分算子研究它们的 Cauchy 型奇异积分的性质，将边值问题转化成积分方程问题，借助于积分方程理论和 Schauder 不动点理论证明了边值问题解的存在性并给出了解的积分表达式.

在文 [23][24] 的基础上，第七章在无界区域上讨论了 Clifford 分析中的双正则函数及其 Plemelj 公式.

最后一章在复 Clifford 代数空间 $\mathcal{A}_n(C)$ 中讨论了复超正则函数的等价条件及性质. 复 Clifford 代数空间的基和实 Clifford 代数空间相同，不同的是系数用复数. 实 Clifford 分析是研究从实向量空间 R^{n+1} 到 $\mathcal{A}_n(R)$ 的函数的数学分支. 复 Clifford 分析是研究从多复向量空间 C^{n+1} 到 $\mathcal{A}_n(C)$ 的函数的数学分支. 对复 Clifford 分析，F.Sommen, J.Ryan 等人做了许多工作，证明了复正则函数在位于复 Clifford 代数中的一个 Lie 群作用下的不变性，并且利用复调和函数去构造全纯函数. 黄沙 [25][26][27] 得到了复正则函数的充分必要条件和复正则函数与复调和函数的关系. 近年来 Clifford 分析中实超正则函数成为学者们研究的热门课题这一，实超正则函数是 Clifford 分析中被研究较多的一类函数，是一类 Clifford 分析中微分方程的解. H 型解是指向量值的超正则函数. W.Hengartner 和 H.Lentwiler 研究了 R^3 中的超正则函数，使 Clifford 分析中的函数理论有了进一步的发展; H.Leutwiler 在文 [28][29][30] 中研究了 H 型解，使超正则函数有了更加具体的表现形式. 本文第八章在复 Clifford 分析中给出了复正则函数和复超正则函数的等价条件，这些条件与单元复分析中全纯函数的 C-R 条件非常类似. 另外还给出了 H 型解和复超正则函数的关系 (见定理 8.3 和 8.4). 这些结论对进一步研究函数的性质有一定的意义.

High Order Singular Integrals and Boundary Value Problems in Clifford Analysis

Abstract

K.T.Vahlen[1] set up Clifford algebra at the begin of last century. It is a kind of algebraic structure which is associative and incommutable. Clifford analysis is a branch of mathematics arising in 1970s. It studies theory of function from real vector space to real incommutable Clifford algebra. It plays an important role in many fields such as other mathematical branches, theory of fields and quantum mechanics and so on. Many mathematicians pay much attention to it. Robert P.Gilbert[2], F.Brack, R.Relanghe, F.Sommen[3], WenGuochun[4], LeHuang Song[5], Huangsha[6−12] and so on did many works through comparing Clifford analysis with single variable, multiple variables complex analysis function theory. The studies to Clifford analysis were developed rapidly by the end of last century.

In this paper, we consider high order singular integrals, function properties and boundary value problems in Clifford analysis. There are eight chapters in this paper.

We discuss high order singular integrals in Clifford analysis from the first chapter to the third chapter. The integral theory is very important in real analysis, it is not only the theory base of modern mathematics, but also one of the main tools in math application. But some practical problems can't be solved by the normal integral. It is necessary to consider singular integrals and even high order singular integrals. Hadamard[13] defined high order singular integrals by the think of the integral for finite part in real analysis. For example, we consider the high order singular integral $\int_a^b \frac{f(x)dx}{(b-x)^{k+\frac{1}{2}}}$, Its finite part is defined like this:

$$F.P.\int_a^b \frac{f(x)dx}{(b-x)^{k+\frac{1}{2}}} = \lim_{\varepsilon\to 0}\left[\int_a^{b-\varepsilon} \frac{f(x)dx}{(b-x)^{k+\frac{1}{2}}} - H(b,k,\varepsilon)\right],$$

Here,

$$H(b,k,\varepsilon) = \sum_{j=0}^{k-1} \frac{f^{(j)}(b)}{j\,!}(x-b)^j.$$

The main idea is to discard the unintegrable part and define the high order singular integral by use of the other parts. In paper[14], Lujianke defined directly high order singular integrals on the closed curve L like this:

$\int_L \frac{f(t)}{(t-t_0)^2}dt = \int_L \frac{f'(t)}{t-t_0}dt$, That is, $\int_L (\frac{-1}{t-t_0})' f(t)dt = \int_L -(\frac{-1}{t-t_0}) f'(t)dt$,Here, comparing $-(\frac{-1}{t-t_0})$ with $(\frac{-1}{t-t_0})'$,the singularity of the former is lower. Professor Lu's theory about high order singular integrals was approved widely in the field of mathematics.Moreover, it is used in practical work. By the think of induction, we use normal integral or weakly-singular integrals to define singular integrals and use the lower order singular integral to define high order singular integrals. Some results about high order singular integrals were proved in the document[14][15][16].

Huangsha has discussed three kinds of high order singular integrals in Clifford analysis. Based on paper[17], by the think of Hadamard principal value of high order singular integrals and the think of induction,we discuss the inductive definition, computation formula, properties and Poincaré-Bertrand exchange formula for high order singular integrals of quasi Bochner-Martinelli- type(in the following, it is abbreviated quasi B-M type) from the first chapter to the third chapter. We know integral operator and integral equation play an important role in solving boundary value problems and differential equations. so the content in this part is significant in mathematical theory and application.

In the fourth chapter, we discuss the solvability of the second kind integral equation and the solution expression formula. The integral equation problem is another important problem in practice. Studying integral equation in Clifford analysis is blank space yet. Resorting to unit resolution ,M. Spivark gave the definition of improper integral defined on the manifold in n dimensional real space in his works <<Calculus on Manifold>>, it can be found in paper[18]. This definition is the generalization of real improper integrals in one dimensional space, it permits that there are many singular points on the manifold. We define improper integrals using the similar method in Clifford analysis. Robert P. Gilbert introduced definition of the commutative factor in paper[2], then he solved partly the incommutable problem of multiplication in Clifford analysis. Resorting to this idea, we design integral operator kernel with commutative factor and study the solvability of the second kind integral equation and series expression formula of solution .

We discuss boundary value problems for the two kinds of functions in the fifth chapter and the sixth chapter. The regular function in Clifford analysis is similar with the holomorphic function in single variable complex analysis,its definition was given firstly in paper[22]. The Cauchy type singular integral and Plemelj formula about regular function have been studied.In paper [20],the author discussed a special boundary value problem for regular function by use of the above results. The definition of bi regular and generalized bi regular functions in Clifford analysis were given in paper [19]. Bi regular functions and generalized bi regular functions are more wider than regular functions. The study to them is a popular branch in the field of function theory in recent years. Enlightened by paper [21],in this paper, the boundary value problem with displacement for these two kinds of functions is discussed in the fifth chapter and the sixth chapter. We design integral operator to study their Cauchy type singular integrals and transform boundary value problems to integral equation problems. Then, resorting to theory of integral equation and theory of Schauder fixed point, we prove the existence of solution to boundary value problems and give integral expression formula of the solution.

Based on paper [23],[24], in the seventh chapter, we discuss the bi regular function and its Plemelj formula on unbounded domain in Clifford analysis.

In the last chapter, we discuss equivalence conditions and properties of complex hypermonogenic functions in complex Clifford algebra space $\mathcal{A}_n(C)$. The base of complex Clifford algebra space is same with real Clifford algebra space, the difference is that the coefficient in complex Clifford algebra space is complex number. Real Clifford analysis is a mathematical branch which studies the function from real vector space R^{n+1} to $\mathcal{A}_n(R)$. Complex Clifford analysis is a mathematical branch which studies the function from complex vector space C^{n+1} to $\mathcal{A}_n(C)$. F. Sommen, J. Ryan and so on did many works about complex Clifford analysis. They proved unchanging property of complex regular functions under a Lie group in complex Clifford algebra and constructed holomorphic functions by use of complex harmonic functions. Huangsha[25][26][27] got sufficient and necessary conditions of complex regular functions and relation between complex regular functions and complex harmonic functions. Recently, the real hypermonogenic function in Clifford analysis is one of the popular topics. People gave many studies to the real hypermonogenic function in Clifford analysis, and it is the solution to a kind of differential equation in Clifford analysis. H type solution is the vector-valued hypermonogenic function. W. Hengartner and H. Lentwiler studied hypermonogenic functions in R^3. So, the theory of function in Clifford analysis was rapidly improved. H.Leutwiler studied H type solution in paper[28][29][30]. So, hypermonogenic functions have more concrete expression . In the eighth chapter, we give equivalence conditions between complex regular functions and hypermonogenic functions in complex Clifford analysis. These conditions are similar with C-R condition of holomorphic functions in single variable complex analysis . In addition, we give the relations between H type solution and complex hypermonogenic functions. It is proved in theorem8.3 and 8.4. These results are significant to study further function properties.

第一章 Clifford 分析中高阶奇异积分的定义

§1.1 引言及预备知识

在单复变中，按 Hadamard[13] 主值的思想，王传荣先生在文 [15] 是用有限部分的思想来定义高阶奇异积分的. 在文 [14] 中，路见可先生直接定义闭曲线 L 上的高阶奇异积分为 $\int_L \frac{f(t)}{(t-t_0)^2}dt = \int_L \frac{f'(t)}{t-t_0}dt$, 即定义 $\int_L (\frac{-1}{t-t_0})' f(t)dt = \int_L (\frac{1}{t-t_0}) f'(t)dt$, 这里 $(\frac{1}{t-t_0})$ 比 $(\frac{-1}{t-t_0})'$ 具有较低阶的奇性. 路先生关于高阶奇异积分的理论得到了数学理论界的广泛认同并在实际工作中得到了应用. 这是按归纳法思想用正常积分或弱奇异积分定义奇异积分，用低阶奇异积分来定义高阶奇异积分. 在多元复分析中人们也讨论了高阶奇异积分的 Hadamard 主值，这方面的有文 [16]. 黄沙在文 [17] 中讨论了 Clifford 分析中 3 种高阶奇异积分，本章在文 [17] 的基础上，借助于高阶奇异积分的 Hadamard 主值的思想以及归纳法的思想，在第一章讨论了几类含一个奇点的拟 Bochner-Martinelli 型 (以下简称为拟 B-M 型) 高阶奇异积分的归纳定义.

记以 $e_0, e_1, e_2, \cdots, e_n$ 为基底的实向量空间为 R^{n+1}, e_0 为 R^{n+1} 的单位元素, R^{n+1} 的元素为

$$x = \sum_{k=0}^{n} x_k e_k, \quad \bar{x} = x_0 e_0 - \sum_{k=1}^{n} x_k e_k,$$

设 $\mathcal{A}$ 是实 Clifford 代数，其基底由

$$e_\phi = e_0,\ e_A : A = (h_1, \cdots, h_r), 1 \le h_1 < h_2 < \cdots < h_r \le n$$

给出，且定义基底元素的乘积适合如下规则

$e_0^2 = 1,\ e_i^2 = -1, i = 1, \cdots, n;\ e_i e_j = -e_j e_i, (i \ne j, i, j = 1, 2, \cdots, n).$

将此乘积运算线性推广到 $\mathcal{A}$, 显然 $\mathcal{A}$ 是不可交换的代数.

$\mathcal{A}$ 中的元素 u 可表示成 $u = \sum_A u_A e_A, u_A \in R$. 定义 $|u|^2 = \Sigma_A |u_A|^2$, 易证

$$|u+v| \le |u| + |v|;\quad |uv| \le 2^{n-1}|u||v|\ ;\ \ ||u| - |v|| \le |u - v|.$$

设 D 为 R^{n+1} 中连通开集，我们研究 $C^m(m\geq 1)$ 类函数集合

$$F_D^{(m)}=\{f|f:D\to\mathcal{A},\ f(x)=\sum_A f_A(x)e_A,\ f_A\in C^m(D)\}.$$

定义 $F_D^{(m)}$ 算子

$$\bar{\partial}_x=e_0\frac{\partial}{\partial x_0}+e_1\frac{\partial}{\partial x_1}+\cdots+e_n\frac{\partial}{\partial x_n};\ \partial_x=e_0\frac{\partial}{\partial x_0}-e_1\frac{\partial}{\partial x_1}-\cdots-e_n\frac{\partial}{\partial x_n}.$$

如果 $\bar{\partial}_x f=0$, 则称 f 为 D 内的左正则函数，简称正则函数.

在下文中，设 Ω 为 D 的边界，$x,y\in\Omega,\bar{\partial}_x f(x,y),\partial_x f(x,y)$ 均为 Hölder 连续函数.

引理 1.1[17] 设 $u=u_0e_0,\ v(x)=\sum\limits_{i=0}^{n}v_i(x)e_i\in C^m(R^{n+1}),m\geq 1,x\in R^{n+1}$, 则

$$\bar{\partial}_x(uv)=(\bar{\partial}_x u)v+u(\bar{\partial}_x v),\tag{1.1}$$

$$\partial_x(uv)=(\partial_x u)v+u(\partial_x v),\tag{1.2}$$

$$u(\bar{\partial}_x\bar{v}+\partial_x v)=-(\bar{\partial}_x u)\bar{v}-(\partial_x u)v+\partial_x(uv)+\bar{\partial}_x(u\bar{v}).\tag{1.3}$$

引理 1.2[17] 设 $x,y\in R^{n+1},x\neq y$, 则

$$\bar{\partial}_x(\bar{x}-\bar{y})=n+1,\quad \partial_x(x-y)=n+1;\tag{1.4}$$

$$\bar{\partial}_x|x-y|^\sigma=\sigma|x-y|^{\sigma-2}(x-y),\qquad \partial_x|x-y|^\sigma=\sigma|x-y|^{\sigma-2}(\bar{x}-\bar{y});\tag{1.5}$$

$$\bar{\partial}_x\left[\frac{\bar{x}-\bar{y}}{(-\alpha)|x-y|^{n+1+\alpha}}\right]=\partial_x\left[\frac{x-y}{(-\alpha)|x-y|^{n+1+\alpha}}\right]=\frac{1}{|x-y|^{n+1+\alpha}};\tag{1.6}$$

$$\begin{aligned}\bar{\partial}_x\left[\frac{1}{(1-n-\alpha)|x-y|^{n-1+\alpha}}\right]&=\frac{x-y}{|x-y|^{n+1+\alpha}},\\ \partial_x\left[\frac{1}{(1-n-\alpha)|x-y|^{n-1+\alpha}}\right]&=\frac{\bar{x}-\bar{y}}{|x-y|^{n+1+\alpha}}.\end{aligned}\tag{1.7}$$

§1.2 含一个奇点的拟 B-M 型高阶奇异积分的归纳定义

黄沙先生在文 [17] 定义了 Clifford 分析中三类高 $\alpha>0$ 阶奇异积分如下

$$\int_\Omega \frac{(\bar{x}-\bar{y})f(x,y)d\sigma_x}{|x-y|^{n+1+\alpha}}=\int_\Omega \frac{f(x,y)(\bar{x}-\bar{y})d\sigma_x}{|x-y|^{n+1+\alpha}}=\int_\Omega \frac{\partial_x f(x,y)d\sigma_x}{(n-1+\alpha)|x-y|^{n-1+\alpha}},\quad y\in\Omega,$$

$$\int_\Omega \frac{(x-y)f(x,y)d\sigma_x}{|x-y|^{n+1+\alpha}}=\int_\Omega \frac{f(x,y)(x-y)d\sigma_x}{|x-y|^{n+1+\alpha}}=\int_\Omega \frac{\bar{\partial}_x f(x,y)d\sigma_x}{(n-1+\alpha)|x-y|^{n-1+\alpha}},\quad y\in\Omega,$$

$$\int_\Omega \frac{f(x,y)d\sigma_x}{|x-y|^{n+1+\alpha}}=\int_\Omega \frac{(\bar{\partial}_x f)(\bar{x}-\bar{y})+(\partial_x f)(x-y)}{2\alpha|x-y|^{n+1+\alpha}}d\sigma_x,\quad y\in\Omega.$$

在此基础上，我们本章按 Hadamard 主值的思想来定义形如

$$\int_\Omega \left[\frac{(\bar{x}-\bar{y})^2}{|x-y|^{n+3+\alpha}}-\frac{(x-y)^2}{|x-y|^{n+3+\alpha}}\right]f(x,y)d\sigma_x,$$

$$\int_\Omega \frac{(\bar{x}-\bar{y})^2 f(x,y)d\sigma_x}{|x-y|^{n+3+\alpha}},\qquad \int_\Omega \frac{(x-y)^2 f(x,y)d\sigma_x}{|x-y|^{n+3+\alpha}}$$

的含一个奇点的拟 B-M 型高阶奇异积分，它们将是研究实分析中高阶奇异积分及积分方程的很有用的工具. 类似于文 [17]，我们是在一个微分公式中去掉高阶奇异部分，用其余部分定义函数关于某种核的积分. 所以首先讨论有关的微分公式.

引理 1.3[17] 设 $x,y\in R^{n+1}$, $x\neq y$, 则

$$\bar{\partial}_x\left(\frac{x-y}{|x-y|^{n+1+\alpha}}\right)=\frac{1-n}{|x-y|^{n+1+\alpha}}-\frac{(n+1+\alpha)(x-y)^2}{|x-y|^{n+\alpha+3}},\tag{1.8}$$

$$\partial_x\left(\frac{\bar{x}-\bar{y}}{|x-y|^{n+1+\alpha}}\right)=\frac{1-n}{|x-y|^{n+1+\alpha}}-\frac{(n+1+\alpha)(\bar{x}-\bar{y})^2}{|x-y|^{n+\alpha+3}}.\tag{1.9}$$

证明 由于 $x,y\in R^{n+1}$, 可设 $x=x_0e_0+x_1e_1+\cdots+x_ne_n$, $y=y_0e_0+y_1e_1+\cdots+y_ne_n$, 则

$$x-y=(x_0-y_0)e_0+(x_1-y_1)e_1+\cdots+(x_n-y_n)e_n,$$

于是

$$\bar{\partial}_x(x-y)=\sum_{k=0}^{n}e_k\frac{\partial(x-y)}{\partial x_k}=1-n.$$

由引理 1.1, 引理 1.2, 知

$$\begin{aligned}\bar{\partial}_x\left(\frac{x-y}{|x-y|^{n+1+\alpha}}\right) &= \frac{\bar{\partial}_x(x-y)}{|x-y|^{n+1+\alpha}}+(x-y)\bar{\partial}_x\left(\frac{1}{|x-y|^{n+1+\alpha}}\right)\\ &= \frac{1-n}{|x-y|^{n+1+\alpha}}-\frac{(n+1+\alpha)(x-y)^2}{|x-y|^{n+3+\alpha}},\end{aligned}$$

(1.8) 式得证. 类似可证 (1.9) 式. **证毕** .

引理 1.4[17] 设 $u(x)=\sum\limits_A u_A(x)e_A$, $v(x)=\sum\limits_{i=0}^{n}v_i(x)e_i\in C^m(R^{n+1})$, $m\geq 1, x\in R^{n+1}$, 记 $()_{x_j}=\frac{\partial()}{\partial_{x_j}}$, 则

$$\bar{\partial}_x(vu)=(\bar{\partial}_x v)u+v(\bar{\partial}_x u)-2\sum_{i,j=1,i\neq j}^{n}v_ie_ie_ju_{x_j},\tag{1.10}$$

$$\partial_x(vu)=(\partial_x v)u+v(\partial_x u)+2\sum_{i,j=1,i\neq j}^{n}v_ie_ie_ju_{x_j}.\tag{1.11}$$

引理 1.5[17] 设 $u(x)=\sum\limits_A u_A(x)e_A$, $v(x)=\sum\limits_{i=0}^{n}v_i(x)e_i\in C^m(R^{n+1})$, $m\geq 1$, $x\in R^{n+1}$, 则

$$[(\partial_x-\bar{\partial}_x)(v+\bar{v})]u=(v+\bar{v})[(\partial_x-\bar{\partial}_x)u]+[(\partial_x-\bar{\partial}_x)(v+\bar{v})]u,\tag{1.12}$$

$$[\partial_x(v+\bar{v})]u=-(v+\bar{v})\partial_x u+\partial_x[(v+\bar{v})u].\tag{1.13}$$

证明 在引理 1.4 中取 v 为 $\bar{v}$, 并且注意到当 $j\geq 1$ 时, $v_j\bar{e}_j=-v_je_j$, 则得

$$\bar{\partial}_x(\bar{v}u)=(\bar{\partial}_x\bar{v})u+\bar{v}(\bar{\partial}_x u)+2\sum_{i,j=1,i\neq j}^{n}v_ie_ie_ju_{x_j},\tag{1.14}$$

$$\partial_x(\bar{v}u)=(\partial_x\bar{v})u+\bar{v}(\partial_x u)-2\sum_{i,j=1,i\neq j}^{n}v_ie_ie_ju_{x_j},\tag{1.15}$$

(1.10) 式加上 (1.14) 式得

$$\bar{\partial}_x[(\bar{v}+v)u] = [\bar{\partial}_x(v+\bar{v})]u + (v+\bar{v})\bar{\partial}_x u, \tag{1.16}$$

(1.11) 式加上 (1.15) 式得

$$\partial_x[(\bar{v}+v)u] = [\partial_x(v+\bar{v})]u + (v+\bar{v})\partial_x u, \tag{1.17}$$

(1.17) 式减 (1.16) 式得 (1.12) 式. 由 (1.17) 式知 $[\partial_x(v+\bar{v})]u = -(v+\bar{v})\partial_x u + \partial_x[(\bar{v}+v)u]$, (1.13) 式得证. **证毕** .

设 $\alpha > 0$, 在引理 1.5 的 (1.12) 式中令 $v = \frac{\bar{x}-\bar{y}}{(-\alpha)|x-y|^{n+1+\alpha}}$, $u = f(x,y)$, 则

$$\left[(\partial_x - \bar{\partial}_x)(\frac{\bar{x}-\bar{y}}{(-\alpha)|x-y|^{n+1+\alpha}} + \frac{x-y}{(-\alpha)|x-y|^{n+1+\alpha}})\right] f(x,y)$$

$$= \frac{1}{-\alpha}\left[\frac{\bar{x}-\bar{y}}{|x-y|^{n+1+\alpha}} + \frac{x-y}{|x-y|^{n+1+\alpha}}\right]\left[(\partial_x - \bar{\partial}_x)f(x,y)\right]$$

$$+\left[(\partial_x - \bar{\partial}_x)\left(\frac{\bar{x}-\bar{y}}{(-\alpha)|x-y|^{n+1+\alpha}} + \frac{x-y}{(-\alpha)|x-y|^{n+1+\alpha}}\right)\right] f(x,y),$$

由引理 1.1, 引理 1.2 可得

$$\left[(\partial_x - \bar{\partial}_x)(\frac{\bar{x}-\bar{y}}{(-\alpha)|x-y|^{n+1+\alpha}} + \frac{x-y}{(-\alpha)|x-y|^{n+1+\alpha}})\right] f(x,y)$$

$$= \Big[\partial_x(\frac{\bar{x}-\bar{y}}{(-\alpha)|x-y|^{n+1+\alpha}}) + \partial_x(\frac{x-y}{(-\alpha)|x-y|^{n+1+\alpha}})$$

$$-\bar{\partial}_x(\frac{\bar{x}-\bar{y}}{(-\alpha)|x-y|^{n+1+\alpha}}) - \bar{\partial}_x(\frac{x-y}{(-\alpha)|x-y|^{n+1+\alpha}})\Big] f(x,y)$$

$$= \left[\frac{\partial_x(\bar{x}-\bar{y})}{(-\alpha)|x-y|^{n+1+\alpha}} + (\bar{x}-\bar{y})\partial_x\left(\frac{1}{(-\alpha)|x-y|^{n+1+\alpha}}\right)\right.$$

$$+\frac{\partial_x(x-y)}{(-\alpha)|x-y|^{n+1+\alpha}} + (x-y)\partial_x\left(\frac{1}{(-\alpha)|x-y|^{n+1+\alpha}}\right)$$

$$-\frac{\bar{\partial}_x(\bar{x}-\bar{y})}{(-\alpha)|x-y|^{n+1+\alpha}} - (\bar{x}-\bar{y})\bar{\partial}_x\left(\frac{1}{(-\alpha)|x-y|^{n+1+\alpha}}\right)$$

$$\left.-\frac{\bar{\partial}_x(x-y)}{(-\alpha)|x-y|^{n+1+\alpha}} -(x-y)\bar{\partial}_x\left(\frac{1}{(-\alpha)|x-y|^{n+1+\alpha}}\right)\right] f(x,y)$$

$$= \left[\frac{1-n}{(-\alpha)|x-y|^{n+1+\alpha}} + \frac{n+1+\alpha}{\alpha}\,\frac{(\bar{x}-\bar{y})^2}{|x-y|^{n+3+\alpha}}\right.$$

$$+\frac{n+1}{(-\alpha)|x-y|^{n+1+\alpha}} + \frac{n+1+\alpha}{\alpha}\,\frac{(x-y)(\bar{x}-\bar{y})}{|x-y|^{n+3+\alpha}}$$

$$-\frac{n+1}{(-\alpha)|x-y|^{n+1+\alpha}} - \frac{n+1+\alpha}{\alpha}\,\frac{(\bar{x}-\bar{y})(x-y)}{|x-y|^{n+3+\alpha}}$$

$$\left.-\frac{1-n}{(-\alpha)|x-y|^{n+1+\alpha}} - \frac{n+1+\alpha}{\alpha}\,\frac{(x-y)^2}{|x-y|^{n+3+\alpha}}\right] f(x,y)$$

$$= \left[\frac{1-n}{(-\alpha)|x-y|^{n+1+\alpha}} + \frac{n+1+\alpha}{\alpha}\,\frac{(\bar{x}-\bar{y})^2}{|x-y|^{n+3+\alpha}}\right.$$

$$+\frac{n+1}{(-\alpha)|x-y|^{n+1+\alpha}} + \frac{n+1+\alpha}{\alpha}\,\frac{|x-y|^2}{|x-y|^{n+3+\alpha}}$$

$$-\frac{n+1}{(-\alpha)|x-y|^{n+1+\alpha}} - \frac{n+1+\alpha}{\alpha}\,\frac{|x-y|^2}{|x-y|^{n+3+\alpha}}$$

$$-\frac{1-n}{(-\alpha)|x-y|^{n+1+\alpha}}-\frac{n+1+\alpha}{\alpha}\frac{(x-y)^2}{|x-y|^{n+3+\alpha}}\Bigg]f(x,y)$$

$$=\frac{n+1+\alpha}{\alpha}\left[\frac{(\bar{x}-\bar{y})^2}{|x-y|^{n+3+\alpha}}-\frac{(x-y)^2}{|x-y|^{n+3+\alpha}}\right]f(x,y),$$

代入上式即得

$$\frac{n+1+\alpha}{\alpha}\left[\frac{(\bar{x}-\bar{y})^2}{|x-y|^{n+3+\alpha}}-\frac{(x-y)^2}{|x-y|^{n+3+\alpha}}\right]f(x,y)$$

$$\begin{aligned}=\;&\frac{1}{\alpha}\left[\frac{\bar{x}-\bar{y}}{|x-y|^{n+1+\alpha}}+\frac{x-y}{|x-y|^{n+1+\alpha}}\right]\left[(\partial_x-\bar{\partial}_x)f(x,y)\right]\\&+(\partial_x-\bar{\partial}_x)\left[(\frac{\bar{x}-\bar{y}}{(-\alpha)|x-y|^{n+1+\alpha}}+\frac{x-y}{(-\alpha)|x-y|^{n+1+\alpha}})f(x,y)\right],\end{aligned}$$

其中左端具有高 $\alpha+1$ 阶奇性，而右端第一项具有 α 阶奇性，右端第二项具有 $\alpha+1$ 阶奇性，所以按 Hadamard 主值的思想，并借助于归纳法思想，我们给出以下的归纳的定义.

定义 1.1

$$\begin{aligned}&\int_\Omega\left[\frac{(\bar{x}-\bar{y})^2}{|x-y|^{n+3+\alpha}}-\frac{(x-y)^2}{|x-y|^{n+3+\alpha}}\right]f(x,y)d\sigma_x\\=\;&\frac{1}{n+1+\alpha}\int_\Omega\frac{(\bar{x}-\bar{y})+(x-y)}{|x-y|^{n+1+\alpha}}\left[(\partial_x-\bar{\partial}_x)f(x,y)\right]d\sigma_x,\end{aligned}\tag{1.18}$$

此处 $\alpha>0,y\in\Omega$.

类似地在引理 1.5 的 (1.12) 式中令 $v=\dfrac{\bar{x}-\bar{y}}{|x-y|^{n+1+\alpha}},u=f(x,y)$, 经过类似计算，可得

$$[\partial_x(v+\bar{v})]\,u=\left[\frac{1-n-\alpha}{|x-y|^{n+1+\alpha}}-\frac{(\bar{x}-\bar{y})^2(n+1+\alpha)}{|x-y|^{n+3+\alpha}}\right]f(x,y)$$

$$-(v+\bar{v})\partial_{x1}u=\frac{-\left[(\bar{x}-\bar{y})+(x-y)\right]}{|x-y|^{n+1+\alpha}}\partial_xf(x,y),$$

由 (1.13) 式得

$$\left[\frac{1-n-\alpha}{|x-y|^{n+1+\alpha}}-\frac{(\bar{x}-\bar{y})^2(n+1+\alpha)}{|x-y|^{n+3+\alpha}}\right]f(x,y)$$

$$=\frac{-[(\bar{x}-\bar{y})+(x-y)]}{|x-y|^{n+1+\alpha}}\partial_x f(x,y)+\partial_x\left[\left(\frac{\bar{x}-\bar{y}}{|x-y|^{n+1+\alpha}}+\frac{x-y}{|x-y|^{n+1+\alpha}}\right)f(x,y)\right],$$

其中左端具有 $\alpha+1$ 阶奇性，右端第一项具有高 α 阶奇性，第二项具有高 $\alpha+1$ 阶奇性，所以我们给出以下的归纳的定义.

定义 1.2

$$\int_\Omega \frac{(\bar{x}-\bar{y})^2 f(x,y)}{|x-y|^{n+3+\alpha}}d\sigma_x$$

$$=\frac{-(n-1+\alpha)}{n+1+\alpha}\int_\Omega \frac{f(x,y)}{|x-y|^{n+1+\alpha}}d\sigma_x+\frac{1}{n+1+\alpha}\int_\Omega \frac{[(\bar{x}-\bar{y})+(x-y)]\,\partial_x f(x,y)}{|x-y|^{n+1+\alpha}}d\sigma_x, \tag{1.19}$$

此处 $\alpha>0, y\in\Omega$. 再利用 (1.18),(1.19) 式类似给出以下定义.

定义 1.3

$$\int_\Omega \frac{(x-y)^2 f(x,y)}{|x-y|^{n+3+\alpha}}d\sigma_x$$

$$=\frac{-(n-1+\alpha)}{n+1+\alpha}\int_\Omega \frac{f(x,y)}{|x-y|^{n+1+\alpha}}d\sigma_x+\frac{1}{n+1+\alpha}\int_\Omega \frac{[(\bar{x}-\bar{y})+(x-y)]\,\bar{\partial}_x f(x,y)}{|x-y|^{n+1+\alpha}}d\sigma_x, \tag{1.20}$$

此处 $\alpha>0, y\in\Omega$.

第二章　含一个奇点的拟 B-M 型高阶奇异积分的计算

以上定义的积分是用已有定义的正常积分或弱奇异积分给出奇异积分的定义，所以这些定义不能用于正常积分．为了进一步研究这种积分的性质以利于在实践中应用，就需要给出将以上积分化成正常积分或弱奇异积分的公式．本章第一节给出了拟 B-M 型高阶奇异积分的递推公式和计算公式，第二节给出了拟 B-M 型高阶奇异积分的微分公式.

§2.1　含一个奇点的拟 B-M 型高阶奇异积分的递推公式和计算公式

若函数 $f(x,y)$, $(x,y)\in \overline{D}_1\times\overline{D}_2$, 经过算子 $\bar{\partial}_x$, ∂_x 运算 $p(p\le m)$ 次都是 Hölder 连续的，并以 $0<\beta<1$ 为 Hölder 指数，则称 f 属于 $H_x^m(\beta)$ 类函数．类似定义 $H_y^p(\beta)$ 类函数．若 $f\in H_x^m(\beta_1)$，同时 $f\in H_y^p(\beta_2)$，则称 f 属于 $H^{m,p}(\beta_1,\beta_2)$ 类函数.

引理 2.1[17] 设 $n>0$, $\alpha>2m>0$, $f(x,y)\in H_x^{2m+2}(\beta)$, $y\in\Omega$, $1\le l\le m$, $l\in N$ 则

$$\int_\Omega \frac{f(x,y)(\bar{x}-\bar{y})}{|x-y|^{n+1+\alpha}}d\sigma_x=\mu\int_\Omega\frac{\Delta_x^l(\partial_x f(x,y))}{|x-y|^{n+\alpha-2l-1}}d\sigma_x, \tag{2.1}$$

$$\int_\Omega \frac{f(x,y)(x-y)}{|x-y|^{n+1+\alpha}}d\sigma_x=\mu\int_\Omega\frac{\Delta_x^l(\bar{\partial}_x f(x,y))}{|x-y|^{n+\alpha-2l-1}}d\sigma_x, \tag{2.2}$$

$$\int_\Omega \frac{f(x,y)}{|x-y|^{n+1+\alpha}}d\sigma_x=\frac{\mu}{\alpha}\int_\Omega\frac{\Delta_x^{l+1}f(x,y)}{|x-y|^{n+\alpha-2l-1}}d\sigma_x, \tag{2.3}$$

此处算子 $\Delta_x=\partial_x\bar{\partial}_x=\bar{\partial}_x\partial_x$, $\mu=\frac{(\alpha-2l-2)!!(n+\alpha-2l-3)!!}{(\alpha-2)!!(n+\alpha-1)!!}$，记号 $r!!$ 表示从 r 开始递减 (每次减 2) 直到最小整数为止所得到数的乘积.

定理 2.1 设 $n>0$, $\alpha>2m>0$, $f(x,y)\in H_x^{2m+2}(\beta)$, $y\in\Omega$, $0<l\le m$, $l\in N$,

则

$$\int_\Omega \frac{(\bar{x}-\bar{y})^2 f(x,y)}{|x-y|^{n+3+\alpha}} d\sigma_x = \frac{1-n}{\alpha} v \int_\Omega \frac{\Delta_x^{l+1} f(x,y)}{|x-y|^{n+\alpha-2l-1}} d\sigma_x + v \int_\Omega \frac{\Delta_x^l \partial_x^2 f(x,y)}{|x-y|^{n+\alpha-2l-1}} d\sigma_x, \tag{2.4}$$

$$\int_\Omega \frac{(x-y)^2 f(x,y)}{|x-y|^{n+3+\alpha}} d\sigma_x = \frac{1-n}{\alpha} v \int_\Omega \frac{\Delta_x^{l+1} f(x,y)}{|x-y|^{n+\alpha-2l-1}} d\sigma_x + v \int_\Omega \frac{\Delta_x^l \bar{\partial}_x^2 f(x,y)}{|x-y|^{n+\alpha-2l-1}} d\sigma_x, \tag{2.5}$$

$$\int_\Omega [\frac{(\bar{x}-\bar{y})^2}{|x-y|^{n+3+\alpha}} - \frac{(x-y)^2}{|x-y|^{n+3+\alpha}}] f(x,y) d\sigma_x = v \int_\Omega \frac{\Delta_x^l (\partial_x^2 - \bar{\partial}_x^2) f(x,y)}{|x-y|^{n+\alpha-2l-1}} d\sigma_x, \tag{2.6}$$

此处 $v = \frac{\mu}{n+1+\alpha}$, μ 如引理 2.1 所述.

证明 利用定义 1.2, 引理 2.1, 有

$$\begin{aligned}
&\int_\Omega \frac{(\bar{x}-\bar{y})^2 f(x,y)}{|x-y|^{n+3+\alpha}} d\sigma_x \\
= &\frac{-(n-1+\alpha)}{n+1+\alpha} \int_\Omega \frac{f(x,y) d\sigma_x}{|x-y|^{n+1+\alpha}} + \frac{1}{n+1+\alpha} \int_\Omega \frac{[(\bar{x}-\bar{y})+(x-y)]\partial_x f(x,y)}{|x-y|^{n+1+\alpha}} d\sigma_x \\
= &\frac{-(n-1+\alpha)}{n+1+\alpha} \int_\Omega \frac{f(x,y)}{|x-y|^{n+1+\alpha}} d\sigma_x + \frac{1}{n+1+\alpha} \int_\Omega \frac{(\bar{x}-\bar{y})\partial_x f(x,y)}{|x-y|^{n+1+\alpha}} d\sigma_x \\
&+ \frac{1}{n+1+\alpha} \int_\Omega \frac{(x-y)\partial_x f(x,y)}{|x-y|^{n+1+\alpha}} d\sigma_x \\
= &\frac{-(n-1+\alpha)}{n+1+\alpha} \frac{\mu}{\alpha} \int_\Omega \frac{\Delta_x^{l+1} f(x,y)}{|x-y|^{n+\alpha-2l-1}} d\sigma_x + \frac{\mu}{n+1+\alpha} \int_\Omega \frac{\Delta_x^l (\partial_x^2 f(x,y))}{|x-y|^{n+\alpha-2l-1}} d\sigma_x \\
&+ \frac{\mu}{n+1+\alpha} \int_\Omega \frac{\Delta_x^{l+1} f(x,y)}{|x-y|^{n+\alpha-2l-1}} d\sigma_x
\end{aligned}$$

$$= \frac{-(n-1)}{\alpha}\frac{\mu}{(n+1+\alpha)}\int_\Omega \frac{\Delta_x^{l+1} f(x,y)}{|x-y|^{n+\alpha-2l-1}}d\sigma_x$$

$$+\frac{\mu}{n+1+\alpha}\int_\Omega \frac{\Delta_x^l(\partial_x^2 f(x,y))}{|x-y|^{n+\alpha-2l-1}}d\sigma_x$$

$$= \frac{-(n-1)}{\alpha}\upsilon\int_\Omega \frac{\Delta_x^{l+1} f(x,y)}{|x-y|^{n+\alpha-2l-1}}d\sigma_x + \upsilon\int_\Omega \frac{\Delta_x^l(\partial_x^2 f(x,y))}{|x-y|^{n+\alpha-2l-1}}d\sigma_x,$$

其中 $\upsilon = \dfrac{\mu}{n+1+\alpha}$, (2.4) 式得证. 类似地, 利用定义 1.3, 引理 2.1, 可得 (2.5) 式. 由 (2.4),(2.5) 式可得

$$\int_\Omega \left[\frac{(\bar{x}-\bar{y})^2}{|x-y|^{n+\alpha+3}} - \frac{(x-y)^2}{|x-y|^{n+\alpha+3}}\right] f(x,y)d\sigma_x$$

$$= \int_\Omega \frac{(\bar{x}-\bar{y})^2 f(x,y)d\sigma_x}{|x-y|^{n+\alpha+3}} - \int_\Omega \frac{(x-y)^2 f(x,y)d\sigma_x}{|x-y|^{n+\alpha+3}}$$

$$= \frac{1-n}{\alpha}\upsilon\int_\Omega \frac{\Delta_x^{l+1} f(x,y)d\sigma_x}{|x-y|^{n+\alpha-2l-1}} + \upsilon\int_\Omega \frac{\Delta_x^l(\partial_x^2 f(x,y))d\sigma_x}{|x-y|^{n+\alpha-2l-1}}$$

$$+\frac{n-1}{\alpha}\upsilon\int_\Omega \frac{\Delta_x^{l+1} f(x,y)d\sigma_x}{|x-y|^{n+\alpha-2l-1}} - \upsilon\int_\Omega \frac{\Delta_x^l(\bar{\partial}_x^2 f(x,y))d\sigma_x}{|x-y|^{n+\alpha-2l-1}}$$

$$= \upsilon\int_\Omega \frac{\Delta_x^l(\partial_x^2 - \bar{\partial}_x^2) f(x,y)}{|x-y|^{n+\alpha-2l-1}}d\sigma_x,$$

(2.6) 式得证. **证毕** .

引理 2.2[17] 设 $f(x,y) \in H_x^{(2k+2)}(\beta)$, $y \in \Omega$, $\lambda_1 = \dfrac{(n-2-r)!!}{(n+2k-r)!!(2k-1-r)!!}$, $0 \le r < 2$, 则下列高阶奇异积分存在并且可以用以下公式计算

$$\int_\Omega \frac{f(x,y)(\bar{x}-\bar{y})d\sigma_x}{|x-y|^{n+2k+2-r}} = \lambda_1 \int_\Omega \frac{(\Delta_x^k \partial_x f(x,y))d\sigma_x}{|x-y|^{n-r}}, \tag{2.7}$$

$$\int_\Omega \frac{f(x,y)(x-y)d\sigma_x}{|x-y|^{n+2k+2-r}} = \lambda_1 \int_\Omega \frac{(\Delta_x^k \bar{\partial}_x f(x,y))d\sigma_x}{|x-y|^{n-r}}, \tag{2.8}$$

$$\int_\Omega \frac{f(x,y)d\sigma_x}{|x-y|^{n+2k+2-r}} = \frac{\lambda_1}{2k+1-r}\int_\Omega \frac{(\Delta_x^{k+1}f(x,y))d\sigma_x}{|x-y|^{n-r}}. \tag{2.9}$$

定理 2.2 设 $f(x,y)\in H_x^{(2k+2)}(\beta), y\in\Omega, 0\le r<2, \lambda_2=\dfrac{(n-2-r)!!}{(n+2k+2-r)!!(2k-1-r)!!}$，则下列高阶奇异积分存在并且可以用以下公式计算

$$\int_\Omega \frac{(\bar{x}-\bar{y})^2 f(x,y)d\sigma_x}{|x-y|^{n+2k+4-r}} = \frac{1-n}{2k+1-r}\lambda_2\int_\Omega \frac{\Delta_x^{k+1}f(x,y)d\sigma_x}{|x-y|^{n-r}} + \lambda_2\int_\Omega \frac{\Delta_x^k\partial_x^2 f(x,y)d\sigma_x}{|x-y|^{n-r}}, \tag{2.10}$$

$$\int_\Omega \frac{(x-y)^2 f(x,y)d\sigma_x}{|x-y|^{n+2k+4-r}} = \frac{1-n}{2k+1-r}\lambda_2\int_\Omega \frac{\Delta_x^{k+1}f(x,y)d\sigma_x}{|x-y|^{n-r}} + \lambda_2\int_\Omega \frac{\Delta_x^k\bar{\partial}_x^2 f(x,y)d\sigma_x}{|x-y|^{n-r}}, \tag{2.11}$$

$$\int_\Omega \left[\frac{(\bar{x}-\bar{y})^2}{|x-y|^{n+2k+4-r}} - \frac{(x-y)^2}{|x-y|^{n+2k+4-r}}\right] f(x,y)d\sigma_x = \lambda_2\int_\Omega \frac{\Delta_x^k(\partial_x^2-\bar{\partial}_x^2)f(x,y)}{|x-y|^{n-r}}d\sigma_x. \tag{2.12}$$

证明 在 (2.4) 式中令 $\alpha=2k+1-r, l=k$, 则

$$\begin{aligned}
v &= \frac{(\alpha-2l-2)!!(n+\alpha-2l-3)!!}{(\alpha-2)!!(n+1+\alpha)!!}\\
&= \frac{(n+\alpha-2l-3)!!}{(n+1+\alpha)!!(\alpha-2)(\alpha-4)...(\alpha-2l)}\\
&= \frac{(n+2k+1-r-2k-3)!!}{(n+2k+2-r)!!(2k+1-r-2)(2k+1-r-4)...(2k+1-r-2k)}\\
&= \frac{(n-2-r)!!}{(n+2k+2-r)!!(2k-1-r)(2k-3-r)...(1-r)}\\
&= \frac{(n-2-r)!!}{(n+2k+2-r)!!(2k-1-r)!!} = \lambda_2,
\end{aligned}$$

这样

$$\int_\Omega \frac{(\bar{x}-\bar{y})^2 f(x,y)d\sigma_x}{|x-y|^{n+2k+4-r}} = \frac{1-n}{2k+1-r}\lambda_2\int_\Omega \frac{\Delta_x^{k+1}f(x,y)d\sigma_x}{|x-y|^{n-r}} + \lambda_2\int_\Omega \frac{\Delta_x^k\partial_x^2 f(x,y)d\sigma_x}{|x-y|^{n-r}},$$

(2.10) 式得证. 类似可证 (2.11) 式. 由 (2.10),(2.11) 式可得

$$\int_\Omega \left[\frac{(\bar{x}-\bar{y})^2}{|x-y|^{n+2k+4-r}} - \frac{(x-y)^2}{|x-y|^{n+2k+4-r}}\right] f(x,y)d\sigma_x$$

$$= \int_\Omega \frac{(\bar{x}-\bar{y})^2}{|x-y|^{n+2k+4-r}} f(x,y)d\sigma_x - \int_\Omega \frac{(x-y)^2}{|x-y|^{n+2k+4-r}} f(x,y)d\sigma_x$$

$$= -\frac{n-1}{2k+1-r}\lambda_2 \int_\Omega \frac{\Delta_x^{k+1} f(x,y)d\sigma_x}{|x-y|^{n-r}} + \lambda_2 \int_\Omega \frac{\Delta_x^k \partial_x^2 f(x,y)d\sigma_x}{|x-y|^{n-r}}$$

$$+\frac{n-1}{2k+1-r}\lambda_2 \int_\Omega \frac{\Delta_x^{k+1} f(x,y)d\sigma_x}{|x-y|^{n-r}} - \lambda_2 \int_\Omega \frac{\Delta_x^k \bar{\partial}_x^2 f(x,y)d\sigma_x}{|x-y|^{n-r}}$$

$$= \lambda_2 \int_\Omega \frac{\Delta_x^k(\partial_x^2 - \bar{\partial}_x^2) f(x,y)}{|x-y|^{n-r}} d\sigma_x,$$

(2.12) 式得证. **证毕** .

例 2.1 设 $f(x,y): R^3 \to \mathcal{A}$, 并且 $f(x,y) \in H^{5,5}(\beta_1,\beta_2)$, 考虑高阶奇异积分

$$\int_\Omega \frac{(x-y)^2 f(x,y)}{|x-y|^7} d\sigma_x.$$

解 在定理 2.2 的 (2.11) 式中令 $n=2,\ \alpha=1$ 则有

$$\int_\Omega \frac{(x-y)^2 f(x,y)}{|x-y|^7} d\sigma_x = \frac{-1}{2}\frac{1}{7!!}\int_\Omega \frac{\Delta_x^2 f(x,y)}{|x-y|} d\sigma_x + \frac{1}{7!!}\int_\Omega \frac{\Delta_x^2 \bar{\partial}_x^2 f(x,y)}{|x-y|} d\sigma_x.$$

§2.2 含一个奇点的拟 B-M 型高阶奇异积分的各种微分公式

引理 2.3[17] 设 $f(x,y) \in H^{(m+2k+2,m)}(\beta_1,\beta_2), 0 < \beta_i < 1, i = 1,2.\ 0 \le r < 2, \lambda_1$ 同引理 2.2 所述，则

$$\partial_y^m \int_\Omega \frac{f(x,y)(\bar{x}-\bar{y})d\sigma_x}{|x-y|^{n+2k+2-r}} = \lambda_1 \int_\Omega \frac{[(\partial_y+\partial_x)^m \Delta_x^k \partial_x f(x,y)]d\sigma_x}{|x-y|^{n-r}}, \tag{2.13}$$

$$\partial_y^m \int_\Omega \frac{f(x,y)(x-y)d\sigma_x}{|x-y|^{n+2k+2-r}} = \lambda_1 \int_\Omega \frac{[(\partial_y+\partial_x)^m \Delta_x^k \bar{\partial}_x f(x,y)]d\sigma_x}{|x-y|^{n-r}}, \tag{2.14}$$

$$\partial_y^m \int_\Omega \frac{f(x,y)d\sigma_x}{|x-y|^{n+2k+2-r}} = \frac{\lambda_1}{2k+1-r}\int_\Omega \frac{[(\partial_y+\partial_x)^m \Delta_x^{k+1} f(x,y)]d\sigma_x}{|x-y|^{n-r}}. \qquad (2.15)$$

定理 2.3 设 $f(x,y) \in H^{(m+2k+2,m)}(\beta_1,\beta_2)$, $0<\beta_i<1, i=1,2$. $0 \le r < 2$, λ_1 同引理 2.2 所述, λ_2 同定理 2.2 所述, 则

$$\partial_y^m \int_\Omega \frac{(\bar{x}-\bar{y})^2 f(x,y)d\sigma_x}{|x-y|^{n+2k+4-r}}$$

$$= \frac{1-n}{(2k+1-r)}\lambda_2 \int_\Omega \frac{(\partial_y+\partial_x)^m \Delta_x^{k+1} f(x,y)}{|x-y|^{n-r}}d\sigma_x + \lambda_2 \int_\Omega \frac{(\partial_y+\partial_x)^m \Delta_x^{k} \partial_x^2 f(x,y)}{|x-y|^{n-r}}d\sigma_x, \qquad (2.16)$$

$$\partial_y^m \int_\Omega \frac{(x-y)^2 f d\sigma_x}{|x-y|^{n+2k+4-r}}$$

$$= \frac{1-n}{(2k+1-r)}\lambda_2 \int_\Omega \frac{(\partial_y+\partial_x)^m \Delta_x^{k+1} f(x,y)}{|x-y|^{n-r}}d\sigma_x + \lambda_2 \int_\Omega \frac{(\partial_y+\partial_x)^m \Delta_x^{k} \bar{\partial}_x^2 f(x,y)}{|x-y|^{n-r}}d\sigma_x, \qquad (2.17)$$

$$\partial_y^m \int_\Omega \left[\frac{(\bar{x}-\bar{y})^2}{|x-y|^{n+2k+4-r}} - \frac{(x-y)^2}{|x-y|^{n+2k+4-r}}\right] f(x,y)d\sigma_x$$

$$= \lambda_2 \int_\Omega \frac{(\partial_y+\partial_x)^m \Delta_x^{k}(\partial_x^2 - \bar{\partial}_x^2) f(x,y)}{|x-y|^{n-r}}d\sigma_x. \qquad (2.18)$$

证明 由定义 1.2, 引理 2.3, 得

$$\partial_y^m \int_\Omega \frac{(\bar{x}-\bar{y})^2 f(x,y)d\sigma_x}{|x-y|^{n+2k+4-r}}$$

$$
\begin{aligned}
&= \frac{-(n+2k-r)}{n+2k+2-r}\partial_y^m \int_\Omega \frac{f(x,y)d\sigma_x}{|x-y|^{n+2k+2-r}} \\
&+\frac{1}{n+2k+2-r}\partial_y^m \int_\Omega \frac{[(\bar{x}-\bar{y})+(x-y)]\partial_x f(x,y)d\sigma_x}{|x-y|^{n+2k+2-r}} \\
&= \frac{-(n+2k-r)}{n+2k+2-r}\partial_y^m \int_\Omega \frac{f(x,y)d\sigma_x}{|x-y|^{n+2k+2-r}} + \frac{1}{n+2k+2-r}\partial_y^m \int_\Omega \frac{(\bar{x}-\bar{y})\partial_x f(x,y)}{|x-y|^{n+2k+2-r}}d\sigma_x \\
&+\frac{1}{n+2k+2-r}\partial_y^m \int_\Omega \frac{(x-y)\partial_x f(x,y)}{|x-y|^{n+2k+2-r}}d\sigma_x \\
&= \frac{-(n+2k-r)}{n+2k+2-r}\partial_y^m \int_\Omega \frac{f(x,y)d\sigma_x}{|x-y|^{n+2k+2-r}} + \frac{1}{n+2k+2-r}\partial_y^m \int_\Omega \frac{(\partial_x f(x,y))(\bar{x}-\bar{y})}{|x-y|^{n+2k+2-r}}d\sigma_x \\
&+\frac{1}{n+2k+2-r}\partial_y^m \int_\Omega \frac{(\partial_x f(x,y))(x-y)}{|x-y|^{n+2k+2-r}}d\sigma_x \\
&= \frac{-(n+2k-r)}{2k+1-r}\lambda_2 \int_\Omega \frac{[(\partial_y+\partial_x)^m \Delta_x^{k+1} f(x,y)]d\sigma_x}{|x-y|^{n-r}} \\
&+\lambda_2 \int_\Omega \frac{[(\partial_y+\partial_x)^m \Delta_x^{k}\partial_x^2 f(x,y)]d\sigma_x}{|x-y|^{n-r}} \\
&+\lambda_2 \int_\Omega \frac{[(\partial_y+\partial_x)^m \Delta_x^{k+1} f(x,y)]d\sigma_x}{|x-y|^{n-r}} \\
&= \frac{1-n}{(2k+1-r)}\lambda_2 \int_\Omega \frac{[(\partial_y+\partial_x)^m \Delta_x^{k+1} f(x,y)]d\sigma_x}{|x-y|^{n-r}} \\
&+\lambda_2 \int_\Omega \frac{[(\partial_y+\partial_x)^m \Delta_x^{k}\partial_x^2 f(x,y)]d\sigma_x}{|x-y|^{n-r}},
\end{aligned}
$$

(2.16) 式得证. 类似地由定义 1.3, 引理 2.3, 可得 (2.17) 式.

由 (2.16) 式，(2.17) 式，得

$$\partial_y^m \int_\Omega \left[\frac{(\bar{x}-\bar{y})^2}{|x-y|^{n+2k+4-r}} - \frac{(x-y)^2}{|x-y|^{n+2k+4-r}} \right] f(x,y) d\sigma_x$$

$$= \partial_y^m \int_\Omega \frac{(\bar{x}-\bar{y})^2 f(x,y)}{|x-y|^{n+2k+4-r}} d\sigma_x - \partial_y^m \int_\Omega \frac{(x-y)^2 f(x,y)}{|x-y|^{n+2k+4-r}} d\sigma_x$$

$$= \frac{1-n}{(n+2k+2-r)(2k+1-r)} \lambda_1 \int_\Omega \frac{[(\partial_y+\partial_x)^m \Delta_x^{k+1} f(x,y)] d\sigma_x}{|x-y|^{n-r}}$$

$$+ \frac{1}{n+2k+2-r} \lambda_1 \int_\Omega \frac{[(\partial_y+\partial_x)^m \Delta_x^k \partial_x^2 f(x,y)] d\sigma_x}{|x-y|^{n-r}}$$

$$- \frac{1-n}{(n+2k+2-r)(2k+1-r)} \lambda_1 \int_\Omega \frac{(\partial_y+\partial_x)^m \Delta_x^{k+1} f(x,y)}{|x-y|^{n-r}} d\sigma_x$$

$$- \frac{1}{n+2k+2-r} \lambda_1 \int_\Omega \frac{(\partial_y+\partial_x)^m \Delta_x^k \bar{\partial}_x^2 f(x,y)}{|x-y|^{n-r}} d\sigma_x$$

$$= \lambda_2 \int_\Omega \frac{(\partial_y+\partial_x)^m \Delta_x^k (\partial_x^2 - \bar{\partial}_x^2) f(x,y)}{|x-y|^{n-r}} d\sigma_x,$$

(2.18) 式得证. **证毕**.

由文 [17] 知

引理 2.4 设 $f(x,y) \in H^{(m+2k+2,m)}(\beta_1,\beta_2), 0 < \beta_i < 1, i = 1,2.$ $0 \le r < 2,$ λ_1 同引理 2.2 所述，则

$$\bar{\partial}_y^m \int_\Omega \frac{f(x,y)(\bar{x}-\bar{y}) d\sigma_x}{|x-y|^{n+2k+2-r}} = \lambda_1 \int_\Omega \frac{[(\bar{\partial}_y+\bar{\partial}_x)^m \Delta_x^k \partial_x f(x,y)] d\sigma_x}{|x-y|^{n-r}}, \tag{2.19}$$

$$\bar{\partial}_y^m \int_\Omega \frac{f(x,y)(x-y) d\sigma_x}{|x-y|^{n+2k+2-r}} = \lambda_1 \int_\Omega \frac{[(\bar{\partial}_y+\bar{\partial}_x)^m \Delta_x^k \bar{\partial}_x f(x,y)] d\sigma_x}{|x-y|^{n-r}}, \tag{2.20}$$

$$\bar{\partial}_y^m \int_\Omega \frac{f(x,y) d\sigma_x}{|x-y|^{n+2k+2-r}} = \frac{\lambda_1}{2k+1-r} \int_\Omega \frac{[(\bar{\partial}_y+\bar{\partial}_x)^m \Delta_x^{k+1} f(x,y)] d\sigma_x}{|x-y|^{n-r}}. \tag{2.21}$$

定理 2.4 设 $f(x,y)\in H^{(m+2k+2,m)}(\beta_1,\beta_2), 0<\beta_i<1, i=1,2.$ $0\leq r<2, \lambda_1$, λ_2 同定理 2.3 所述，则

$$
\begin{aligned}
&\bar\partial_y^m\int_\Omega\frac{(\bar x-\bar y)^2f(x,y)d\sigma_x}{|x-y|^{n+2k+4-r}}\\
=\;&\frac{1-n}{(2k+1-r)}\lambda_2\int_\Omega\frac{(\bar\partial_y+\bar\partial_x)^m\Delta_x^{k+1}f}{|x-y|^{n-r}}d\sigma_x \qquad (2.22)\\
&+\lambda_2\int_\Omega\frac{(\bar\partial_y+\bar\partial_x)^m\Delta_x^k\partial_x^2f}{|x-y|^{n-r}}d\sigma_x,
\end{aligned}
$$

$$
\begin{aligned}
&\bar\partial_y^m\int_\Omega\frac{(x-y)^2fd\sigma_x}{|x-y|^{n+2k+3-r}}\\
=\;&\frac{2-n}{(2k+1-r)}\lambda_2\int_\Omega\frac{(\bar\partial_y+\bar\partial_x)^m\Delta_x^{k+1}f}{|x-y|^{n-1-r}}d\sigma_x \qquad (2.23)\\
&+\lambda_2\int_\Omega\frac{(\bar\partial_y+\bar\partial_x)^m\Delta_x^k\bar\partial_x^2f}{|x-y|^{n-1-r}}d\sigma_x,
\end{aligned}
$$

$$
\begin{aligned}
&\bar\partial_y^m\int_\Omega\left[\frac{(\bar x-\bar y)^2}{|x-y|^{n+2k+3-r}}-\frac{(x-y)^2}{|x-y|^{n+2k+3-r}}\right]fd\sigma_x \qquad (2.24)\\
=\;&\frac{1}{n+2k+1-r}\lambda_1\int_\Omega\frac{(\bar\partial_y+\bar\partial_x)^m\Delta_x^k(\partial_x^2-\bar\partial_x^2)f}{|x-y|^{n-1-r}}d\sigma_x.
\end{aligned}
$$

证明 由定义 1.2, 引理 2.4, 得

$$
\begin{aligned}
&\bar{\partial}_y^m \int_\Omega \frac{(\bar{x}-\bar{y})^2 f(x,y)d\sigma_x}{|x-y|^{n+2k+4-r}}\\
= \ & \frac{-(n+2k-r)}{n+2k+2-r}\bar{\partial}_y^m \int_\Omega \frac{f(x,y)d\sigma_x}{|x-y|^{n+2k+2-r}}\\
&+\frac{1}{n+2k+2-r}\bar{\partial}_y^m \int_\Omega \frac{[(\bar{x}-\bar{y})+(x-y)]\partial_x f(x,y)d\sigma_x}{|x-y|^{n+2k+2-r}}\\
= \ & \frac{-(n+2k-r)}{n+2k+1-r}\bar{\partial}_y^m \int_\Omega \frac{f(x,y)d\sigma_x}{|x-y|^{n+2k+2-r}}+\frac{1}{n+2k+2-r}\bar{\partial}_y^m \int_\Omega \frac{(\bar{x}-\bar{y})\partial_x f(x,y)}{|x-y|^{n+2k+2-r}}d\sigma_x\\
&+\frac{1}{n+2k+2-r}\bar{\partial}_y^m \int_\Omega \frac{(x-y)\partial_x f(x,y)}{|x-y|^{n+2k+2-r}}d\sigma_x\\
= \ & \frac{-(n+2k-r)}{n+2k+1-r}\bar{\partial}_y^m \int_\Omega \frac{f(x,y)d\sigma_x}{|x-y|^{n+2k+2-r}}+\frac{1}{n+2k+2-r}\bar{\partial}_y^m \int_\Omega \frac{(\partial_x f(x,y))(\bar{x}-\bar{y})}{|x-y|^{n+2k+2-r}}d\sigma_x\\
&+\frac{1}{n+2k+2-r}\bar{\partial}_y^m \int_\Omega \frac{(\partial_x f(x,y))(x-y)}{|x-y|^{n+2k+2-r}}d\sigma_x\\
= \ & \frac{-(n+2k-r)}{n+2k+2-r}\frac{\lambda_1}{2k+1-r}\int_\Omega \frac{[(\bar{\partial}_y+\bar{\partial}_x)^m\Delta_x^{k+1}f(x,y)]d\sigma_x}{|x-y|^{n-r}}\\
&+\frac{1}{n+2k+2-r}\lambda_1\int_\Omega \frac{[(\bar{\partial}_y+\bar{\partial}_x)^m\Delta_x^{k}\partial_x^2 f(x,y)]d\sigma_x}{|x-y|^{n-r}}\\
&+\frac{1}{n+2k+2-r}\lambda_1\int_\Omega \frac{[(\bar{\partial}_y+\bar{\partial}_x)^m\Delta_x^{k+1} f(x,y)]d\sigma_x}{|x-y|^{n-r}}\\
= \ & \frac{1-n}{(2k+2-r)}\lambda_2\int_\Omega \frac{[(\bar{\partial}_y+\bar{\partial}_x)^m\Delta_x^{k+1} f(x,y)]d\sigma_x}{|x-y|^{n-r}}\\
&+\lambda_2\int_\Omega \frac{[(\bar{\partial}_y+\bar{\partial}_x)^m\Delta_x^{k}\partial_x^2 f(x,y)]d\sigma_x}{|x-y|^{n-r}},
\end{aligned}
$$

(2.22) 式得证. 类似地由定义 1.3, 引理 2.4, 可得 (2.23) 式.

由 (2.22) 式, (2.23) 式可得

$$\bar{\partial}_y^m \int_\Omega [\frac{(\bar{x}-\bar{y})^2}{|x-y|^{n+2k+3-r}} - \frac{(x-y)^2}{|x-y|^{n+2k+3-r}}] f d\sigma_x$$

$$= \bar{\partial}_y^m \int_\Omega \frac{(\bar{x}-\bar{y})^2 f}{|x-y|^{n+2k+3-r}} d\sigma_x - \bar{\partial}_y^m \int_\Omega \frac{(x-y)^2 f}{|x-y|^{n+2k+3-r}} d\sigma_x$$

$$= \frac{2-n}{(n+2k+1-r)(2k+1-r)} \lambda_1 \int_\Omega \frac{(\bar{\partial}_y+\bar{\partial}_x)^m \Delta_x^{k+1} f}{|x-y|^{n-1-r}} d\sigma_x$$

$$+ \frac{1}{n+2k+1-r} \lambda_1 \int_\Omega \frac{(\bar{\partial}_y+\bar{\partial}_x)^m \Delta_x^k \partial_x^2 f}{|x-y|^{n-1-r}} d\sigma_x$$

$$- \frac{2-n}{(n+2k+1-r)(2k+1-r)} \lambda_1 \int_\Omega \frac{(\bar{\partial}_y+\bar{\partial}_x)^m \Delta_x^{k+1} f}{|x-y|^{n-1-r}} d\sigma_x$$

$$- \frac{1}{n+2k+1-r} \lambda_1 \int_\Omega \frac{(\bar{\partial}_y+\bar{\partial}_x)^m \Delta_x^k \bar{\partial}_x^2 f}{|x-y|^{n-1-r}} d\sigma_x$$

$$= \lambda_2 \int_\Omega \frac{(\bar{\partial}_y+\bar{\partial}_x)^m \Delta_x^k (\partial_x^2 - \bar{\partial}_x^2) f}{|x-y|^{n-1-r}} d\sigma_x,$$

(2.24) 式得证. **证毕** .

引理 2.5 设 $f(x,y) \in H^{(m+2k+2,m)}(\beta_1,\beta_2), 0<\beta_i<1, i=1,2.\ 0 \le r < 2, \lambda_1, \lambda_2$ 同定理 2.3 所述, 则

$$\bar{\partial}_y^m \partial_y^p \int_\Omega \frac{f(x,y)(\bar{x}-\bar{y}) d\sigma_x}{|x-y|^{n+2k+2-r}} = \lambda_1 \int_\Omega \frac{[(\bar{\partial}_y+\bar{\partial}_x)^m (\partial_y+\partial_x)^p \Delta_x^k \partial_x f(x,y)] d\sigma_x}{|x-y|^{n-r}}, \quad (2.25)$$

$$\bar{\partial}_y^m \partial_y^p \int_\Omega \frac{f(x,y)(x-y) d\sigma_x}{|x-y|^{n+2k+2-r}} = \lambda_1 \int_\Omega \frac{[(\bar{\partial}_y+\bar{\partial}_x)^m (\partial_y+\partial_x)^p \Delta_x^k \bar{\partial}_x f(x,y)] d\sigma_x}{|x-y|^{n-r}}, \quad (2.26)$$

$$\bar{\partial}_y^m \partial_y^p \int_\Omega \frac{f(x,y)d\sigma_x}{|x-y|^{n+2k+2-r}} = \frac{\lambda_1}{2k+1-r}\int_\Omega \frac{[(\bar{\partial}_y+\bar{\partial}_x)^m(\partial_y+\partial_x)^p\Delta_x^{k+1}f(x,y)]d\sigma_x}{|x-y|^{n-r}}. \tag{2.27}$$

定理 2.5 设 $f(x,y)\in H^{(m+2k+2,m)}(\beta_1,\beta_2), 0<\beta_i<1, i=1,2.\ 0\le r<2, \lambda_1$ 同引理 2.2 所述，则

$$\begin{aligned} &\bar{\partial}_y^m \partial_y^p \int_\Omega \frac{(\bar{x}-\bar{y})^2 f(x,y)d\sigma_x}{|x-y|^{n+2k+4-r}} \\ =\ & \frac{1-n}{(2k+1-r)}\lambda_2\int_\Omega \frac{(\bar{\partial}_y+\bar{\partial}_x)^m(\partial_y+\partial_x)^p\Delta_x^{k+1}f(x,y)}{|x-y|^{n-r}}d\sigma_x \\ &+\lambda_2\int_\Omega \frac{(\bar{\partial}_y+\bar{\partial}_x)^m(\partial_y+\partial_x)^p\Delta_x^k\partial_x^2 f(x,y)}{|x-y|^{n-r}}d\sigma_x, \end{aligned} \tag{2.28}$$

$$\begin{aligned} &\bar{\partial}_y^m \partial_y^p \int_\Omega \frac{(x-y)^2 f(x,y)}{|x-y|^{n+2k+4-r}}d\sigma_x \\ =\ & \frac{1-n}{(2k+1-r)}\lambda_2\int_\Omega \frac{(\bar{\partial}_y+\bar{\partial}_x)^m(\partial_y+\partial_x)^p\Delta_x^{k+1}f(x,y)}{|x-y|^{n-r}}d\sigma_x \\ &+\lambda_2\int_\Omega \frac{(\bar{\partial}_y+\bar{\partial}_x)^m(\partial_y+\partial_x)^p\Delta_x^k\bar{\partial}_x^2 f(x,y)}{|x-y|^{n-r}}d\sigma_x, \end{aligned} \tag{2.29}$$

$$\begin{aligned} &\bar{\partial}_y^m \partial_y^p \int_\Omega \left[\frac{(\bar{x}-\bar{y})^2}{|x-y|^{n+2k+4-r}} - \frac{(x-y)^2}{|x-y|^{n+2k+4-r}}\right]f(x,y)d\sigma_x \\ =\ & \lambda_2\int_\Omega \frac{(\bar{\partial}_y+\bar{\partial}_x)^m(\partial_y+\partial_x)^p\Delta_x^k(\partial_x^2-\bar{\partial}_x^2)f(x,y)}{|x-y|^{n-r}}d\sigma_x. \end{aligned} \tag{2.30}$$

证明 由定义 1.2, 引理 2.5, 得

$$\bar{\partial}_y^m \partial_y^p \int_\Omega \frac{(\bar{x}-\bar{y})^2 f(x,y) d\sigma_x}{|x-y|^{n+2k+4-r}}$$

$$= \frac{-(n+2k-r)}{n+2k+2-r} \bar{\partial}_y^m \partial_y^p \int_\Omega \frac{f(x,y)}{|x-y|^{n+2k+2-r}} d\sigma_x$$

$$+ \frac{1}{n+2k+2-r} \bar{\partial}_y^m \partial_y^p \int_\Omega \frac{[(\bar{x}-\bar{y})+(x-y)]\partial_x f(x,y)}{|x-y|^{n+2k+2-r}} d\sigma_x$$

$$= \frac{-(n+2k-r)}{2k+1-r} \lambda_2 \int_\Omega \frac{[(\bar{\partial}_y+\bar{\partial}_x)^m(\partial_y+\partial_x)^p \Delta_x^{k+1} f(x,y)] d\sigma_x}{|x-y|^{n-r}}$$

$$+ \lambda_2 \int_\Omega \frac{[(\bar{\partial}_y+\bar{\partial}_x)^m(\partial_y+\partial_x)^p \Delta_x^k \partial_x^2 f(x,y)] d\sigma_x}{|x-y|^{n-r}}$$

$$+ \lambda_2 \int_\Omega \frac{[(\bar{\partial}_y+\bar{\partial}_x)^m(\partial_y+\partial_x)^p \Delta_x^{k+1} f(x,y)] d\sigma_x}{|x-y|^{n-r}}$$

$$= \frac{1-n}{(2k+1-r)} \lambda_2 \int_\Omega \frac{(\bar{\partial}_y+\bar{\partial}_x)^m(\partial_y+\partial_x)^p \Delta_x^{k+1} f(x,y)}{|x-y|^{n-r}} d\sigma_x$$

$$+ \lambda_2 \int_\Omega \frac{(\bar{\partial}_y+\bar{\partial}_x)^m(\partial_y+\partial_x)^p \Delta_x^k \partial_x^2 f(x,y)}{|x-y|^{n-r}} d\sigma_x,$$

(2.28) 式得证.

类似地由定义 1.3, 引理 2.5, 可证 (2.29) 式.

由 (2.28) 式, (2.29) 式可得

$$\bar{\partial}_y^m \partial_y^p \int_\Omega \left[\frac{(\bar{x}-\bar{y})^2}{|x-y|^{n+2k+4-r}} - \frac{(x-y)^2}{|x-y|^{n+2k+4-r}} \right] f(x,y) d\sigma_x$$

$$= \bar{\partial}_y^m \partial_y^p \int_\Omega \frac{(\bar{x}-\bar{y})^2 f}{|x-y|^{n+2k+4-r}} d\sigma_x - \bar{\partial}_y^m \partial_y^p \int_\Omega \frac{(x-y)^2 f(x,y)}{|x-y|^{n+2k+4-r}} d\sigma_x$$

$$= \frac{1-n}{(n+2k+4-r)(2k+1-r)} \lambda_1 \int_\Omega \frac{(\bar{\partial}_y+\bar{\partial}_x)^m (\partial_y+\partial_x)^p \Delta_x^{k+1} f(x,y)}{|x-y|^{n-r}} d\sigma_x$$

$$+\lambda_2 \int_\Omega \frac{(\bar{\partial}_y+\bar{\partial}_x)^m (\partial_y+\partial_x)^p \Delta_x^k \partial_x^2 f(x,y)}{|x-y|^{n-r}} d\sigma_x$$

$$-\frac{1-n}{(2k+1-r)} \lambda_2 \int_\Omega \frac{(\bar{\partial}_y+\bar{\partial}_x)^m (\partial_y+\partial_x)^p \Delta_x^{k+1} f(x,y)}{|x-y|^{n-1-r}} d\sigma_x$$

$$-\lambda_2 \int_\Omega \frac{(\bar{\partial}_y+\bar{\partial}_x)^m (\partial_y+\partial_x)^p \Delta_x^k \bar{\partial}_x^2 f(x,y)}{|x-y|^{n-r}} d\sigma_x$$

$$= \lambda_2 \int_\Omega \frac{(\bar{\partial}_y+\bar{\partial}_x)^m (\partial_y+\partial_x)^p \Delta_x^k (\partial_x^2 - \bar{\partial}_x^2) f(x,y)}{|x-y|^{n-r}} d\sigma_x,$$

(2.30) 式得证. **证毕**.

由文 [17] 知

引理 2.6 设 $f(x,y) \in H^{(m+2k+2,m)}(\beta_1,\beta_2)$, $0<\beta_i<1$, $i=1,2$, $t_1,t_2 \in \Omega$. $0 \le r < 2$, λ_1 同引理 2.2 所述, 则

$$\bar{\partial}_{t_1}^m \partial_{t_2}^p \int_\Omega \frac{f(x,t_2)(\bar{x}-\bar{t}_1) d\sigma_x}{|x-t_1|^{n+2k+2-r}} = \lambda_1 \int_\Omega \frac{[(\bar{\partial}_x^m \partial_{t_2}^p \Delta_x^k \partial_x f(x,t_2)] d\sigma_x}{|x-t_1|^{n-r}}, \tag{2.31}$$

$$\bar{\partial}_{t_1}^m \partial_{t_2}^p \int_\Omega \frac{f(x,t_2)(x-t_1) d\sigma_x}{|x-t_1|^{n+2k+2-r}} = \lambda_1 \int_\Omega \frac{[(\bar{\partial}_x^m \partial_{t_2}^p \Delta_x^k \bar{\partial}_x f(x,t_2)] d\sigma_x}{|x-t_1|^{n-r}}, \tag{2.32}$$

$$\bar{\partial}_{t_1}^m \partial_{t_2}^p \int_\Omega \frac{f(x,t_2) d\sigma_x}{|x-t_1|^{n+2k+2-r}} = \frac{\lambda_1}{2k+1-r} \int_\Omega \frac{[(\bar{\partial}_x^m \partial_{t_2}^p \Delta_x^{k+1} f(x,t_2)] d\sigma_x}{|x-t_1|^{n-r}}. \tag{2.33}$$

定理 2.6 设 $f(x,y) \in H^{(m+2k+2,m)}(\beta_1,\beta_2)$, $0<\beta_i<1$, $i=1,2$, $t_1,t_2 \in \Omega$. $0 \le r <$

2, λ_2 同定理 2.2 所述, 则

$$\begin{aligned}&\bar{\partial}_{t_1}^m \partial_{t_2}^p \int_\Omega \frac{(\bar{x}-\bar{t}_1)^2 f(x,t_2)}{|x-t_1|^{n+2k+4-r}} d\sigma_x \\ =& \frac{\lambda_2(1-n)}{2k+1-r}\int_\Omega \frac{\bar{\partial}_x^m \partial_{t_2}^p \Delta_x^{k+1} f(x,t_2)}{|x-t_1|^{n-r}} d\sigma_x + \lambda_2 \int_\Omega \frac{\bar{\partial}_x^m \partial_{t_2}^p \Delta_x^k \partial_x^2 f(x,t_2)}{|x-t_1|^{n-r}} d\sigma_x,\end{aligned} \tag{2.34}$$

$$\begin{aligned}&\bar{\partial}_{t_1}^m \partial_{t_2}^p \int_\Omega \frac{(x-t_1)^2 f(x,t_2)}{|x-t_1|^{n+2k+4-r}} d\sigma_x \\ =& \frac{\lambda_2(1-n)}{2k+1-r}\int_\Omega \frac{\bar{\partial}_x^m \partial_{t_2}^p \Delta_x^{k+1} f(x,t_2)}{|x-t_1|^{n-r}} d\sigma_x + \lambda_2 \int_\Omega \frac{\bar{\partial}_x^m \partial_{t_2}^p \Delta_x^k \bar{\partial}_x^2 f(x,t_2)}{|x-t_1|^{n-r}} d\sigma_x,\end{aligned} \tag{2.35}$$

$$\begin{aligned}&\bar{\partial}_{t_1}^m \partial_{t_2}^p \int_\Omega \left[\frac{(\bar{x}-\bar{t}_1)^2}{|x-t_1|^{n+2k+4-r}} - \frac{(x-t_1)^2}{|x-t_1|^{n+2k+4-r}}\right] f(x,y) d\sigma_x \\ =& \lambda_2 \int_\Omega \frac{\bar{\partial}_x^m \partial_{t_2}^p \Delta_x^k (\partial_x^2 - \bar{\partial}_x^2) f(x,t_2)}{|x-t_1|^{n-r}} d\sigma_x.\end{aligned} \tag{2.36}$$

证明 由定理 2.2, 引理 1.1, 引理 1.2, 知

$$\begin{aligned}&\bar{\partial}_{t_1}^m \partial_{t_2}^p \int_\Omega \frac{(\bar{x}-\bar{t}_1)^2 f(x,t_2)}{|x-t_1|^{n+2k+4-r}} d\sigma_x \\ =& \bar{\partial}_{t_1}^m \partial_{t_2}^p \left(\frac{1-n}{2k+1-r}\lambda_2 \int_\Omega \frac{\Delta_x^{k+1} f(x,t_2) d\sigma_x}{|x-t_1|^{n-r}} + \lambda_2 \int_\Omega \frac{\Delta_x^k \partial_x^2 f(x,t_2) d\sigma_x}{|x-t_1|^{n-r}}\right) \\ =& \frac{\lambda_2(1-n)}{2k+1-r}\bar{\partial}_{t_1}^m \int_\Omega \frac{\partial_{t_2}^p \Delta_x^{k+1} f(x,t_2) d\sigma_x}{|x-t_1|^{n-r}} + \lambda_2 \bar{\partial}_{t_1}^m \int_\Omega \frac{\partial_{t_2}^p \Delta_x^k \partial_x^2 f(x,t_2) d\sigma_x}{|x-t_1|^{n-r}}\end{aligned}$$

$$= \frac{\lambda_2(1-n)}{2k+1-r}\bar{\partial}_{t_1}^{m-1}\int_\Omega \frac{[\partial_{t_2}^p\Delta_x^{k+1}f(x,t_2)](x-t_1)(n-r)}{|x-t_1|^{n+2-r}}d\sigma_x$$

$$+\lambda_2\bar{\partial}_{t_1}^{m-1}\int_\Omega \frac{[\partial_{t_2}^p\Delta_x^k\partial_x^2 f(x,t_2)](x-t_1)(n-r)}{|x-t_1|^{n+2-r}}d\sigma_x$$

$$= \frac{\lambda_2(1-n)}{2k+1-r}\bar{\partial}_{t_1}^{m-1}\int_\Omega \frac{\bar{\partial}_x\partial_{t_2}^p\Delta_x^{k+1}f(x,t_2)}{|x-t_1|^{n-r}}d\sigma_x+\lambda_2\bar{\partial}_{t_1}^{m-1}\int_\Omega \frac{\bar{\partial}_x\partial_{t_2}^p\Delta_x^k\partial_x^2 f(x,t_2)}{|x-t_1|^{n-r}}d\sigma_x$$

$$= \frac{\lambda_2(1-n)}{2k+1-r}\int_\Omega \frac{\bar{\partial}_x^m\partial_{t_2}^p\Delta_x^{k+1}f(x,t_2)}{|x-t_1|^{n-r}}d\sigma_x+\lambda_2\int_\Omega \frac{\bar{\partial}_x^m\partial_{t_2}^p\Delta_x^k\partial_x^2 f(x,t_2)}{|x-t_1|^{n-r}}d\sigma_x,$$

(2.34) 式得证. 类似地，由定理 2.2, 引理 1.1, 引理 1.2, 可得 (2.35) 式.

由 (2.34) 式，(2.35) 式可得

$$\bar{\partial}_{t_1}^m\partial_{t_2}^p\int_\Omega\left[\frac{(\bar{x}-\bar{t}_1)^2}{|x-t_1|^{n+2k+4-r}}-\frac{(x-t_1)^2}{|x-t_1|^{n+2k+4-r}}\right]f(x,y)d\sigma_x$$

$$= \bar{\partial}_{t_1}^m\partial_{t_2}^p\int_\Omega\frac{(\bar{x}-\bar{t}_1)^2}{|x-t_1|^{n+2k+4-r}}f(x,y)d\sigma_x-\bar{\partial}_{t_1}^m\partial_{t_2}^p\int_\Omega\frac{(x-t_1)^2}{|x-t_1|^{n+2k+4-r}}f(x,y)d\sigma_x$$

$$= \frac{\lambda_2(1-n)}{2k+1-r}\int_\Omega \frac{\bar{\partial}_x^m\partial_{t_2}^p\Delta_x^{k+1}f(x,t_2)}{|x-t_1|^{n-r}}d\sigma_x+\lambda_2\int_\Omega \frac{\bar{\partial}_x^m\partial_{t_2}^p\Delta_x^k\partial_x^2 f(x,t_2)}{|x-t_1|^{n-r}}d\sigma_x$$

$$+\frac{\lambda_2(n-1)}{2k+1-r}\int_\Omega \frac{\bar{\partial}_x^m\partial_{t_2}^p\Delta_x^{k+1}f(x,t_2)}{|x-t_1|^{n-r}}d\sigma_x-\lambda_2\int_\Omega \frac{\bar{\partial}_x^m\partial_{t_2}^p\Delta_x^k\bar{\partial}_x^2 f(x,t_2)}{|x-t_1|^{n-r}}d\sigma_x$$

$$= \lambda_2\int_\Omega \frac{\bar{\partial}_x^m\partial_{t_2}^p\Delta_x^k(\partial_x^2-\bar{\partial}_x^2)f(x,t_2)}{|x-t_1|^{n-r}}d\sigma_x,$$

(2.36) 式得证. **证毕** .

第三章　含两个奇点的拟 B-M 型高阶奇异积分

在实际工作中有时一个问题可能会涉及多个奇点，另外在研究积分算子、积分方程时也经常会需要考虑含两个奇点的积分问题，所以本章讨论含有两个奇点的拟 B-M 型高阶奇异积分的归纳定义，计算公式，微分公式等.

§3.1　含两个奇点的拟 B-M 型高阶奇异积分的归纳定义及计算公式

如前所述，我们仍设 Ω 为 D 的边界， $x \neq t$, $x,t \in \Omega$. 利用 D 的可分性，设区域 $D^x \cap D^t = \emptyset$, $\bar{D^x} \cup \bar{D^t} = \bar{D}$, D^x, D^t 的边界分别为 Ω^x, Ω^t, 且适合 $x \in \Omega^x$, $t \in \Omega^t$. 又设 Ω^x, Ω^t 的定向均与 Ω 的定向协调. 且 $\Omega, \Omega^x, \Omega^t$ 的定向分别是 D, D^x, D^t 诱导定向. 再设 $\bar{D^x} \cap \bar{D^t} = \Sigma$, 且对任何 $y \in \Sigma$ 均有 $|y-x| = |y-t|$. 记以 x, t 为奇点的积分核分别为 $K^x(x,y), K^t(x,y)$. 借助于普通积分的可加性思想，我们给出含二个奇点的奇异积分定义.

定义 3.1　设 $\varphi(x,y) \in H^{(m,p)}(\gamma_1,\gamma_2), 0<\gamma_i<1, i=1,2$. 我们定义在 Ω 上含有两个奇点的奇异积分为

$$\int_\Omega \frac{\varphi(x,y)}{|y-t|^{n+1+\alpha}|x-y|^{n+1+\beta}} d\sigma_y$$

$$= \int_{\Omega^t} \frac{\varphi(x,y)}{|y-t|^{n+1+\alpha}|x-y|^{n+1+\beta}} d\sigma_y + \int_{\Omega^x} \frac{\varphi(x,y)}{|y-t|^{n+1+\alpha}|x-y|^{n+1+\beta}} d\sigma_y,$$

此处 $\alpha, \beta > 0, x, t \in \Omega$. m, p 由两个核奇性的阶数来定.

以下定理给出了含两个奇点的拟 B-M 型高阶奇异积分的递推公式.

定理 3.1　设 $\varphi(x,y) \in H^{(m,p)}(\gamma_1,\gamma_2)$, $\dfrac{\varphi(x,y)}{|y-t|^{n+\alpha}} \in H_y^{2m_1+2}(\gamma_3)$, $\dfrac{\varphi(x,y)}{|x-y|^{n+\beta}} \in H_y^{2m_2+2}(\gamma_4), 0<\gamma_i<1$, $i=1,2,3,4$. $\alpha > 2m_1 > 0$, $\beta > 2m_2 > 0$, $1 < l_1 \le m_1$, $1 < l_2 \le m_2$, l_1, $l_2 \in N$, 而且

$$\mu_1=\frac{(\alpha-2l_1-2)!!(n+\alpha-2l_1-1)!!}{(\alpha-2)!!(n+\alpha-1)!!},\quad \mu_2=\frac{(\beta-2l_2-2)!!(n+\beta-2l_2-1)!!}{(\beta-2)!!(n+\beta-1)!!},$$

则

$$\begin{aligned}&\int_\Omega\frac{\varphi(x,y)}{|y-t|^{n+1+\alpha}|x-y|^{n+1+\beta}}d\sigma_y\\ =&\ \frac{\mu_1}{\alpha(n+\alpha-2l_1-1)}\int_{\Omega^t}\frac{\Delta_y^{l_1+1}\frac{\varphi(x,y)}{|x-y|^{n+1+\beta}}}{|y-t|^{n+\alpha-2l_1-1}}d\sigma_y\\ &+\frac{\mu_2}{\beta(n+\beta-2l_2-1)}\int_{\Omega^x}\frac{\Delta_y^{l_2+1}\frac{\varphi(x,y)}{|y-t|^{n+1+\alpha}}}{|x-y|^{n+\beta-2l_2-1}}d\sigma_y.\end{aligned}\tag{3.1}$$

证明 由定义 3.1, 引理 2.1, 得

$$\begin{aligned}&\int_\Omega\frac{\varphi(x,y)}{|y-t|^{n+1+\alpha}|x-y|^{n+1+\beta}}d\sigma_y\\ =&\ \int_{\Omega^t}\frac{\frac{\varphi(x,y)}{|x-y|^{n+1+\beta}}}{|y-t|^{n+1+\alpha}}d\sigma_y+\int_{\Omega^x}\frac{\frac{\varphi(x,y)}{|y-t|^{n+1+\alpha}}}{|x-y|^{n+1+\beta}}d\sigma_y\\ =&\ \frac{\mu_1}{\alpha(n+\alpha-2l_1-1)}\int_{\Omega^t}\frac{\Delta_y^{l_1+1}\frac{\varphi(x,y)}{|x-y|^{n+1+\beta}}}{|y-t|^{n+\alpha-2l_1-1}}d\sigma_y\\ &+\frac{\mu_2}{\beta(n+\beta-2l_2-1)}\int_{\Omega^x}\frac{\Delta_y^{l_2+1}\frac{\varphi(x,y)}{|y-t|^{n+1+\alpha}}}{|x-y|^{n+\beta-2l_2-1}}d\sigma_y.\end{aligned}$$

证毕 .

推论 3.1 设 $\varphi(x,y)\in H^{(m,p)}(\gamma_1,\gamma_2)$, $\frac{\varphi(x,y)}{|y-t|^{n+1+\alpha}}$, $\frac{\varphi(x,y)}{|x-y|^{n+1+\alpha}}\in H_y^{2m+2}(\gamma_3)$, $0<\gamma_i<1$, $i=1,2,3$. $0<\alpha<2m, 0<l\le m$, $l\in N$, 而且

$$\mu_1=\frac{(\alpha-2l_1-2)!!(n+\alpha-2l_1-1)!!}{(\alpha-2)!!(n+\alpha-1)!!}$$

如定理 3.1 所述，则

$$\int_{\Omega}\frac{\varphi(x,y)}{|y-t|^{n+1+\alpha}|x-y|^{n+1+\alpha}}d\sigma_y$$

$$=\frac{\mu_1}{\alpha(n+\alpha-2l_1-1)}\left(\int_{\Omega^t}\frac{\Delta_y^{l_1+1}\frac{\varphi(x,y)}{|x-y|^{n+1+\alpha}}}{|y-t|^{n+\alpha-2l_1-1}}d\sigma_y+\int_{\Omega^x}\frac{\Delta_y^{l_1+1}\frac{\varphi(x,y)}{|y-t|^{n+1+\alpha}}}{|x-y|^{n+\alpha-2l_1-1}}d\sigma_y\right). \tag{3.2}$$

证明 在定理 3.1 的证明中令 $\alpha=\beta$, 这时 $m_1=m_2, l_1=l_2$, 易得推论 3.1.证毕 .

下面的定理 3.2 给出了含两个奇点的拟 B-M 型高阶奇异积分的计算公式.

定理 3.2 设 $\frac{\varphi(x,y)}{|y-t|^{n+2k_1+2-r_1}}\in H_y^{2k_1+2}(\gamma_3)$, $\frac{\varphi(x,y)}{|x-y|^{n+2k_2+2-r_2}}\in H_y^{2k_2+2}(\gamma_4)$, $0<\gamma_i<1,\ i=3,4.\ 0\le r_j<2, j=1,2.$ 而且

$$\lambda_3=\frac{(n-2-r_1)!!}{(n+2k_1-r_1)!!(2k_1+1-r_1)!!},\quad \lambda_4=\frac{(n-2-r_2)!!}{(n+2k_2-r_2)!!(2k_2+1-r_2)!!},$$

则

$$\int_{\Omega}\frac{\varphi(x,y)}{|y-t|^{n+2k_1+2-r_1}|x-y|^{n+2k_2+2-r_2}}d\sigma_y$$

$$=\lambda_3\int_{\Omega^t}\frac{\Delta_y^{k_1+1}\frac{\varphi(x,y)}{|x-y|^{n+2k_2+2-r_2}}}{|y-t|^{n-r_1}}d\sigma_y+\lambda_4\int_{\Omega^x}\frac{\Delta_y^{k_2+1}\frac{\varphi(x,y)}{|y-t|^{n+2k_1+2-r_1}}}{|x-y|^{n-r_2}}d\sigma_y. \tag{3.3}$$

证明 在定理 3.1 中令 $\alpha=2k_1+1-r_1,\beta=2k_2+1-r_2,l_1=k_1,l_2=k_2$, 由引理 2.2, 得

$$\int_{\Omega}\frac{\varphi(x,y)}{|y-t|^{n+2k_1+2-r_1}|x-y|^{n+2k_2+2-r_2}}d\sigma_y$$

$$=\int_{\Omega^t}\frac{\frac{\varphi(x,y)}{|x-y|^{n+2k_2+2-r_2}}}{|y-t|^{n+2k_1+2-r_1}}d\sigma_y+\int_{\Omega^x}\frac{\frac{\varphi(x,y)}{|y-t|^{n+2k_1+2-r_1}}}{|x-y|^{n+2k_2+2-r_2}}d\sigma_y$$

$$=\lambda_3\int_{\Omega^t}\frac{\Delta_y^{k_1+1}\frac{\varphi(x,y)}{|x-y|^{n+2k_2+2-r_2}}}{|y-t|^{n-r_1}}d\sigma_y+\lambda_4\int_{\Omega^x}\frac{\Delta_y^{k_2+1}\frac{\varphi(x,y)}{|y-t|^{n+2k_1+2-r_1}}}{|x-y|^{n-r_2}}d\sigma_y,$$

由文献 [7] 知定理 3.2 中等式右端的积分在普通意义下收敛，所以上述高阶奇异积分存在，且定理 3.2 给出了其计算公式. **证毕** .

推论 3.2 设 $\frac{\varphi(x,y)}{|y-t|^{n+2k_1+2-r_1}}$, $\frac{\varphi(x,y)}{|x-y|^{n+2k_1+2-r_1}} \in H_y^{2k_1+2}(\gamma)$, $0<\gamma<1$, $0 \le r_1 < 2$, λ_3 如定理 3.2 所述，则

$$\int_\Omega \frac{\varphi(x,y)}{|y-t|^{n+2k_1+2-r_1}|x-y|^{n+2k_1+2-r_1}} d\sigma_y$$
$$= \lambda_3 \left(\int_{\Omega^t} \frac{\Delta_y^{k_1+1} \frac{\varphi(x,y)}{|x-y|^{n+2k_1+2-r_1}}}{|y-t|^{n-r_1}} d\sigma_y + \int_{\Omega^x} \frac{\Delta_y^{k_1+1} \frac{\varphi(x,y)}{|y-t|^{n+2k_1+2-r_1}}}{|x-y|^{n-r_1}} d\sigma_y \right). \tag{3.4}$$

证明 在定理 3.2 的证明中令 $k_1 = k_2, l_1 = l_2$, 这时 $\lambda_3 = \lambda_4$, 易得推论 3.2. **证毕** .

§3.2 含两个奇点的拟 B-M 型高阶奇异积分的微分公式

定理3.3 设 $\frac{\varphi(x,y)}{|y-t|^{n+2k_1+2-r_1}} \in H_{y,t}^{(2k_2+2+m,m)}(\gamma_1,\ \gamma_2)$, $\frac{\varphi(x,y)}{|x-y|^{n+2k_2+2-r_2}} \in H_y^{2k_1+2+m}(\gamma_3)$,

$0<\gamma_i<1$, $i=1,2,3$. $0 \le r_j < 2$, $j=1,2$. λ_3, λ_4 如定理 3.2 所述，则

$$\partial_t^m \int_\Omega \frac{\varphi(x,y)}{|y-t|^{n+2k_1+2-r_1}|x-y|^{n+2k_2+2-r_2}} d\sigma_y$$
$$= \lambda_3 \int_{\Omega_t} \frac{\partial_y^m \Delta_y^{k_1+1} \frac{\varphi(x,y)}{|x-y|^{n+2k_2+2-r_2}}}{|y-t|^{n-r_1}} d\sigma_y + \lambda_4 \int_{\Omega_x} \frac{\partial_t^m \Delta_y^{k_2+1} \frac{\varphi(x,y)}{|y-t|^{n+2k_1+2-r_1}}}{|x-y|^{n-r_2}} d\sigma_y, \tag{3.5}$$

$$\partial_x^m \int_\Omega \frac{\varphi(x,y)}{|y-t|^{n+2k_1+2-r_1}|x-y|^{n+2k_2+2-r_2}} d\sigma_y$$
$$= \lambda_3 \int_{\Omega_t} \frac{\partial_x^m \Delta_y^{k_1+1} \frac{\varphi(x,y)}{|x-y|^{n+2k_2+2-r_2}}}{|y-t|^{n-r_1}} d\sigma_y + \lambda_4 \int_{\Omega_x} \frac{(\partial_y+\partial_x)^m \Delta_y^{k_2+1} \frac{\varphi(x,y)}{|y-t|^{n+2k_1+2-r_1}}}{|x-y|^{n-r_2}} d\sigma_y. \tag{3.6}$$

证明 由定义 3.1, (2.9) 式， (2.15) 式，得

$$\partial_t^m \int_\Omega \frac{\varphi(x,y)}{|y-t|^{n+2k_1+2-r_1}|x-y|^{n+2k_2+2-r_2}} d\sigma_y$$

$$= \partial_t^m \int_{\Omega^t} \frac{\frac{\varphi(x,y)}{|x-y|^{n+2k_2+2-r_2}}}{|y-t|^{n+2k_1+2-r_1}} d\sigma_y + \partial_t^m \int_{\Omega^x} \frac{\frac{\varphi(x,y)}{|y-t|^{n+2k_1+2-r_1}}}{|x-y|^{n+2k_2+2-r_2}} d\sigma_y$$

$$= \lambda_3 \int_{\Omega^t} \frac{(\partial_y+\partial_t)^m \Delta_y^{k_1+1} \frac{\varphi(x,y)}{|x-y|^{n+2k_2+2-r_2}}}{|y-t|^{n-r_1}} d\sigma_y$$

$$+\lambda_4 \partial_t^m \int_{\Omega^x} \frac{\Delta_y^{k_2+1} \frac{\varphi(x,y)}{|y-t|^{n+2k_1+2-r_1}}}{|x-y|^{n-r_2}} d\sigma_y$$

$$= \lambda_3 \int_{\Omega^t} \frac{\partial_y^m \Delta_y^{k_1+1} \frac{\varphi(x,y)}{|x-y|^{n+2k_2+2-r_2}}}{|y-t|^{n-r_1}} d\sigma_y + \lambda_4 \int_{\Omega^x} \frac{\partial_t^m \Delta_y^{k_2+1} \frac{\varphi(x,y)}{|y-t|^{n+2k_1+2-r_1}}}{|x-y|^{n-r_2}} d\sigma_y,$$

(3.5) 式得证. 类似地由定义 3.1, (2.9) 式, (2,15) 式得 (3.5) 式. 证毕 .

定理 3.4 设 $\frac{\varphi(x,y)}{|y-t|^{n+2k_1+2-r_1}} \in H_{y,t}^{(2k_2+2+m,m)}(\gamma_1,\gamma_2)$, $\frac{\varphi(x,y)}{|x-y|^{n+2k_2+2-r_2}} \in H_y^{(2k_1+2+m)}(\gamma_3)$, $0<\gamma_i<1, i=1,2,3.$ $0\le r_j<2, j=1,2.$ λ_3,λ_4 如定理 3.2 所述, 则

$$\bar\partial_t^m \int_\Omega \frac{\varphi(x,y)}{|y-t|^{n+2k_1+2-r_1}|x-y|^{n+2k_2+2-r_2}} d\sigma_y$$

$$= \lambda_3 \int_{\Omega_t} \frac{\bar\partial_y^m \Delta_y^{k_1+1} \frac{\varphi(x,y)}{|x-y|^{n+2k_2+2-r_2}}}{|y-t|^{n-r_1}} d\sigma_y + \lambda_4 \int_{\Omega_x} \frac{\bar\partial_t^m \Delta_y^{k_2+1} \frac{\varphi(x,y)}{|y-t|^{n+2k_1+2-r_1}}}{|x-y|^{n-r_2}} d\sigma_y, \tag{3.7}$$

$$\begin{aligned}
&\bar{\partial}_x^m \int_\Omega \frac{\varphi(x,y)}{|y-t|^{n+2k_1+2-r_1}|x-y|^{n+2k_2+2-r_2}} d\sigma_y \\
=\ & \lambda_3 \int_{\Omega_t} \frac{\bar{\partial}_x^m \Delta_y^{k_1+1} \frac{\varphi(x,y)}{|x-y|^{n+2k_2+2-r_2}}}{|y-t|^{n-r_1}} d\sigma_y + \lambda_4 \int_{\Omega_x} \frac{(\bar{\partial}_y + \bar{\partial}_x)^m \Delta_y^{k_2+1} \frac{\varphi(x,y)}{|y-t|^{n+2k_1+2-r_1}}}{|x-y|^{n-r_2}} d\sigma_y.
\end{aligned} \tag{3.8}$$

证明 由定义 3.1, 引理 1.1, (2.9), (2.21) 式, 得

$$\begin{aligned}
&\bar{\partial}_t^m \int_\Omega \frac{\varphi(x,y)}{|y-t|^{n+2k_1+2-r_1}|x-y|^{n+2k_2+2-r_2}} d\sigma_y \\
=\ & \bar{\partial}_t^m \int_{\Omega^t} \frac{\frac{\varphi(x,y)}{|x-y|^{n+2k_2+2-r_2}}}{|y-t|^{n+2k_1+2-r_1}} d\sigma_y + \bar{\partial}_t^m \int_{\Omega^x} \frac{\frac{\varphi(x,y)}{|y-t|^{n+2k_1+2-r_1}}}{|x-y|^{n+2k_2+2-r_2}} d\sigma_y \\
=\ & \lambda_3 \int_{\Omega_t} \frac{(\bar{\partial}_y + \bar{\partial}_t)^m \Delta_y^{k_1+1} \frac{\varphi(x,y)}{|x-y|^{n+2k_2+2-r_2}}}{|y-t|^{n-r_1}} d\sigma_y + \lambda_4 \bar{\partial}_t^m \int_{\Omega_x} \frac{\Delta_y^{k_2+1} \frac{\varphi(x,y)}{|y-t|^{n+2k_1+2-r_1}}}{|x-y|^{n-r_2}} d\sigma_y \\
=\ & \lambda_3 \int_{\Omega_t} \frac{\bar{\partial}_y^m \Delta_y^{k_1+1} \frac{\varphi(x,y)}{|x-y|^{n+2k_2+2-r_2}}}{|y-t|^{n-r_1}} d\sigma_y + \lambda_4 \int_{\Omega_x} \frac{\bar{\partial}_t^m \Delta_y^{k_2+1} \frac{\varphi(x,y)}{|y-t|^{n+2k_1+2-r_1}}}{|x-y|^{n-r_2}} d\sigma_y,
\end{aligned}$$

(3.7) 式得证. 类似地, 由定义 3.1, 引理 1.1, (1.19), (2.21) 式得 (3.8) 式. **证毕** .

定理 3.5 设 $\frac{\varphi(x,y)}{|y-t|^{n+2k_1+2-r_1}} \in H_{y,t}^{(2k_2+2+m,m)}(\gamma_1,\gamma_2)$, $\frac{\varphi(x,y)}{|x-y|^{n+2k_2+2-r_2}} \in$ $H_y^{(2k_1+2+m)}(\gamma_3)$, $0 < \gamma_i < 1, i = 1,2,3.$ $0 \le r_j < 2, j = 1,2.$ λ_3, λ_4 如定理 3.2 所

述，则

$$\bar{\partial}_t^m \partial_t^p \int_\Omega \frac{\varphi(x,y)}{|y-t|^{n+2k_1+2-r_1}|x-y|^{n+2k_2+2-r_2}} d\sigma_y$$

$$= \lambda_3 \int_{\Omega_t} \frac{\bar{\partial}_y^m \partial_y^p \Delta_y^{k_1+1} \frac{\varphi(x,y)}{|x-y|^{n+2k_2+2-r_2}}}{|y-t|^{n-r_1}} d\sigma_y \tag{3.9}$$

$$+\lambda_4 \int_{\Omega_x} \frac{\bar{\partial}_t^m \partial_t^p \Delta_y^{k_2+1} \frac{\varphi(x,y)}{|y-t|^{n+2k_1+2-r_1}}}{|x-y|^{n-r_2}} d\sigma_y,$$

$$\bar{\partial}_x^m \partial_x^p \int_\Omega \frac{\varphi(x,y)}{|y-t|^{n+2k_1+2-r_1}|x-y|^{n+2k_2+2-r_2}} d\sigma_y$$

$$= \lambda_3 \int_{\Omega_t} \frac{\bar{\partial}_x^m \partial_x^p \Delta_y^{k_1+1} \frac{\varphi(x,y)}{|x-y|^{n+2k_2+2-r_2}}}{|y-t|^{n-r_1}} d\sigma_y \tag{3.10}$$

$$+\lambda_4 \int_{\Omega_x} \frac{(\bar{\partial}_y+\bar{\partial}_x)^m (\partial_y+\partial_x)^p \Delta_y^{k_2+1} \frac{\varphi(x,y)}{|y-t|^{n+2k_1+2-r_1}}}{|x-y|^{n-r_2}} d\sigma_y.$$

证明 由定义 3.1, 引理 1.1, (2.9), (2.27) 式，得

$$\bar{\partial}_t^m \partial_t^p \int_\Omega \frac{\varphi(x,y)}{|y-t|^{n+2k_1+2-r_1}|x-y|^{n+2k_2+2-r_2}} d\sigma_y$$

$$= \bar{\partial}_t^m \partial_t^p \int_{\Omega^t} \frac{\frac{\varphi(x,y)}{|x-y|^{n+2k_2+2-r_2}}}{|y-t|^{n+2k_1+2-r_1}} d\sigma_y + \bar{\partial}_t^m \partial_t^p \int_{\Omega^x} \frac{\frac{\varphi(x,y)}{|y-t|^{n+2k_1+2-r_1}}}{|x-y|^{n+2k_2+2-r_2}} d\sigma_y$$

$$= \lambda_3 \int_{\Omega_t} \frac{(\bar{\partial}_y+\bar{\partial}_t)^m (\partial_y+\partial_t)^p \Delta_y^{k_1+1} \frac{\varphi(x,y)}{|x-y|^{n+2k_2+2-r_2}}}{|y-t|^{n-r_1}} d\sigma_y$$

$$+\lambda_4 \bar{\partial}_t^m \partial_t^p \int_{\Omega_x} \frac{\Delta_y^{k_2+1} \frac{\varphi(x,y)}{|y-t|^{n+2k_1+2-r_1}}}{|x-y|^{n-r_2}} d\sigma_y$$

$$= \lambda_3 \int_{\Omega^t} \frac{\bar{\partial}_y^m \partial_y^p \Delta_y^{k_1+1} \frac{\varphi(x,y)}{|x-y|^{n+2k_2+2-r_2}}}{|y-t|^{n-r_1}} d\sigma_y + \lambda_4 \int_{\Omega_x} \frac{\bar{\partial}_t^m \partial_t^p \Delta_y^{k_2+1} \frac{\varphi(x,y)}{|y-t|^{n+2k_1+2-r_1}}}{|x-y|^{n-r_2}} d\sigma_y,$$

(3.9) 式得证. 类似地由定义 3.1, 引理 1.1,(2.9),(2.27) 式得 (3.10) 式. **证毕** .

定理 3.6 设 $\frac{\varphi(x,y)}{|y-t|^{n+2k_1+2-r_1}} \in H_{y,t}^{(2k_2+2+m,m)}(\gamma_1,\gamma_2)$, $\frac{\varphi(x,y)}{|x-y|^{n+2k_2+2-r_2}} \in H_y^{(2k_1+2+m)}(\gamma_3)$, $0<\gamma_i<1, i=1,2,3,\ 0\le r_j<2, j=1,2$. λ_3,λ_4, 如定理 3.2 所述, 则

$$\bar{\partial}_{t_1}^m \partial_{t_2}^p \int_\Omega \frac{\varphi(t_2,y)}{|y-t_2|^{n+2k_1+2-r_1}|t_1-y|^{n+2k_2+2-r_2}} d\sigma_y$$

$$= \lambda_3 \int_{\Omega^{t_2}} \frac{\bar{\partial}_{t_1}^m (\partial_y+\partial_{t_2})^p \Delta_y^{k_1+1} \frac{\varphi(t_2,y)}{|t_1-y|^{n+2k_2+2-r_2}}}{|y-t_2|^{n-r_1}} d\sigma_y \tag{3.11}$$

$$+\lambda_4 \int_{\Omega^{t_1}} \frac{\bar{\partial}_y^m \partial_{t_2}^p \Delta_y^{k_2+1} \frac{\varphi(x,y)}{|y-t_2|^{n+2k_1+2-r_1}}}{|y-t_1|^{n-r_2}} d\sigma_y.$$

证明 由定义 3.1, 引理 1.1,(2.15) 式, (2.33) 式, 得

$$\bar{\partial}_{t_1}^m \partial_{t_2}^p \int_\Omega \frac{\varphi(t_2,y)}{|y-t_2|^{n+2k_1+2-r_1}|t_1-y|^{n+2k_2+2-r_2}} d\sigma_y$$

$$= \bar{\partial}_{t_1}^m \partial_{t_2}^p \int_{\Omega^{t_1}} \frac{\frac{\varphi(t_2,y)}{|y-t_2|^{n+2k_1+2-r_1}}}{|t_1-y|^{n+2k_2+2-r_2}} d\sigma_y + \bar{\partial}_{t_1}^m \partial_{t_2}^p \int_{\Omega^{t_2}} \frac{\frac{\varphi(t_2,y)}{|t_1-y|^{n+2k_2+2-r_2}}}{|y-t_2|^{n+2k_1+2-r_1}} d\sigma_y$$

$$= \lambda_3 \bar{\partial}_{t_1}^m \int_{\Omega^{t_2}} \frac{(\partial_y+\partial_{t_2})^p \Delta_y^{k_1+1} \frac{\varphi(t_2,y)}{|t_1-y|^{n+2k_2+2-r_2}}}{|y-t_2|^{n-r_1}} d\sigma_y + \lambda_4 \int_{\Omega^{t_1}} \frac{\bar{\partial}_y^m \partial_{t_2}^p \Delta_y^{k_2+1} \frac{\varphi(x,y)}{|y-t_2|^{n+2k_1+2-r_1}}}{|y-t_1|^{n-r_2}} d\sigma_y$$

$$= \lambda_3 \int_{\Omega^{t_2}} \frac{\bar{\partial}_{t_1}^m (\partial_y+\partial_{t_2})^p \Delta_y^{k_1+1} \frac{\varphi(t_2,y)}{|t_1-y|^{n+2k_2+2-r_2}}}{|y-t_2|^{n-r_1}} d\sigma_y + \lambda_4 \int_{\Omega^{t_1}} \frac{\bar{\partial}_y^m \partial_{t_2}^p \Delta_y^{k_2+1} \frac{\varphi(x,y)}{|y-t_2|^{n+2k_1+2-r_1}}}{|y-t_1|^{n-r_2}} d\sigma_y.$$

证毕 .

§3.3 含两个奇点的拟 B-M 型高阶奇异积分的 Poincaré-Bertrand 置换公式

在一般情况下，积分次序的交换问题要求条件并不高，但是由于 Clifford 数乘法没有交换律，从而函数的乘法也没有交换律，所以积分次序的交换问题也需要认真讨论. 黄沙教授在文献 [10] 中给出了在 Cauchy 主值意义下的积分的 Poincaré-Bertrand 置换公式，这为解积分方程和讨论积分算子的性质打下了较好的基础. 对于高阶奇异积分讨论 Poincaré-Bertrand 置换公式，一样是非常重要的.

引理 3.1 说明在一定的条件下，封闭曲面上的弱奇异积分是可以交换次序的.

引理 3.1 设 $\varphi(x,y)\in H^{(2,\,2)}(\beta_1,\beta_2)$, $0<\beta_i<1$, $i=1,2$, $r>0$, $h>0$, 且 $n>r+h$, x, $t\in\Omega$, 则

$$\int_\Omega\left[\int_\Omega\frac{\varphi(x,y)}{|y-t|^{n-r}|x-y|^{n-h}}d\sigma_y\right]d\sigma_x=\int_\Omega\left[\frac{1}{|y-t|^{n-r}}\int_\Omega\frac{\varphi(x,y)d\sigma_x}{|x-y|^{n-h}}\right]d\sigma_y. \quad (3.12)$$

证明 由 Hadamard 定理可知当 $n>r+h$ 时，有 $\left|\int_\Omega\frac{\varphi(x,y)}{|y-t|^{n-r}|x-y|^{n-h}}d\sigma_y\right|\leq\frac{G}{|x-t|^{n-(r+h)}}$，这表明 (3.12) 式左端具有弱奇性，所以 (3.12) 式是普通积分，由 Stokes 定理 $\int_\Omega fd\sigma_xg=\int_D[(f\bar\partial)g+f(\bar\partial g)]d\sigma_x$ 得

$$\begin{aligned}&\int_\Omega\left[\int_\Omega\frac{\varphi(x,y)}{|y-t|^{n-r}|x-y|^{n-h}}d\sigma_y\right]d\sigma_x\\=\;&\int_\Omega\left[(\int_D\frac{\varphi(x,y)}{|y-t|^{n-r}|x-y|^{n-h}|}\bar\partial_y)dy\right]d\sigma_x\\=\;&\int_D\left[(\int_D\frac{\varphi(x,y)}{|y-t|^{n-r}|x-y|^{n-h}}\bar\partial_y)dy\right]\bar\partial_xdx\\=\;&\int_D\int_D(\frac{\varphi(x,y)}{|y-t|^{n-r}|x-y|^{n-h}}\bar\partial_y)\bar\partial_xdydx,\end{aligned}$$

又

$$\int_{\Omega}\left[\frac{1}{|y-t|^{n-r}}\int_{\Omega}\frac{\varphi(x,y)d\sigma_x}{|x-y|^{n-h}}\right]d\sigma_y$$

$$= \int_{\Omega}\frac{1}{|y-t|^{n-r}}(\int_{D}\frac{\varphi(x,y)}{|x-y|^{n-h}}\bar{\partial}_x dx)d\sigma_y$$

$$= \int_{D}(\frac{1}{|y-t|^{n-r}}\int_{D}\frac{\varphi(x,y)}{|x-y|^{n-h}}\bar{\partial}_x)dx\bar{\partial}_y dy$$

$$= \int_{D}\int_{D}(\frac{\varphi(x,y)}{|y-t|^{n-r}|x-y|^{n-h}}\bar{\partial}_x)\bar{\partial}_y dxdy,$$

由题设函数 $\frac{\varphi(x,y)}{|y-t|^{n-r}|x-y|^{n-h}}$ 在区域 D 内有连续的二阶偏导数，所以 $\bar{\partial}_x,\bar{\partial}_y$ 可交换顺序，而 $d\sigma_x$，$d\sigma_y$ 是体积微元，可任意交换次序，所以上面两式右端相等，这样，

$$\int_{\Omega}\left[\int_{\Omega}\frac{\varphi(x,y)}{|y-t|^{n-r}|x-y|^{n-h}}d\sigma_y\right]d\sigma_x = \int_{\Omega}\left[\frac{1}{|y-t|^{n-r}}\int_{\Omega}\frac{\varphi(x,y)d\sigma_x}{|x-y|^{n-h}}\right]d\sigma_y.$$

证毕．

定理 3.7 设 $\varphi(x,y)\in H^{(4,4)}(\beta_1,\beta_2)$，$0<\beta_i<1$，$r$，$h>\beta_i$，$i=1,2$，$n>1$，$0\le r<2, n>h+r, x,t\in\Omega$, 则

$$\begin{aligned}
&\int_{\Omega}\frac{y-t}{|y-t|^{n+2-r}}\int_{\Omega}\frac{\varphi(x,y)d\sigma_x}{|x-y|^{n-h}}d\sigma_y \\
=& \int_{\Omega}\left[\int_{\Omega}\frac{(y-t)\varphi(x,y)}{|y-t|^{n+2-r}|x-y|^{n-h}}d\sigma_y\right]d\sigma_x \\
&+\int_{\Omega}\left[\int_{\Omega^t}\frac{1}{(n-r)|y-t|^{n-r}}\bar{\partial}_x(\frac{\varphi(x,y)}{|x-y|^{n-h}})d\sigma_y\right]d\sigma_x \\
&+\int_{\Omega}\left[\int_{\Omega^x}\frac{(\bar{\partial}_x+\bar{\partial}_y)}{(n-r)|x-y|^{n-h}}(\frac{\varphi(x,y)}{|y-t|^{n-r}})d\sigma_y\right]d\sigma_x,
\end{aligned} \tag{3.13}$$

$$\int_\Omega \frac{\bar{y}-\bar{t}}{|y-t|^{n+2-r}} \int_\Omega \frac{\varphi(x,y)d\sigma_x}{|x-y|^{n-h}} d\sigma_y$$

$$= \int_\Omega \left[\int_\Omega \frac{(\bar{y}-\bar{t})\varphi(x,y)}{|y-t|^{n+2-r}|x-y|^{n-h}} d\sigma_y\right] d\sigma_x$$

$$+ \int_\Omega \left[\int_{\Omega^t} \frac{1}{(n-r)|y-t|^{n-r}} \partial_x (\frac{\varphi(x,y)}{|x-y|^{n-h}}) d\sigma_y\right] d\sigma_x \tag{3.14}$$

$$+ \int_\Omega \left[\int_{\Omega^x} \frac{\partial_x + \partial_y}{(n-r)|x-y|^{n-h}} (\frac{\varphi(x,y)}{|y-t|^{n-r}}) d\sigma_y\right] d\sigma_x.$$

证明 在题设条件下，利用引理 1.1, 引理 1.2, 和引理 3.1 可得

$$\int_\Omega \frac{y-t}{|y-t|^{n+2-r}} \int_\Omega \frac{\varphi(x,y)d\sigma_x}{|x-y|^{n-h}} d\sigma_y$$

$$= \int_\Omega \frac{1}{(n-r)|y-t|^{n-r}} \left[\int_\Omega \bar{\partial}_y \frac{\varphi(x,y)}{|x-y|^{n-h}} d\sigma_x\right] d\sigma_y$$

$$= \int_\Omega \frac{1}{(n-r)|y-t|^{n-r}} \left[\int_\Omega \frac{\bar{\partial}_y \varphi(x,y)}{|x-y|^{n-h}} + \frac{\varphi(x,y)(x-y)(n-h)}{|x-y|^{n+2-h}} d\sigma_x\right] d\sigma_y$$

$$= \int_\Omega \left[\int_\Omega \frac{\bar{\partial}_y \varphi(x,y)}{(n-r)|y-t|^{n-r}|x-y|^{n-h}} d\sigma_y\right] d\sigma_x$$

$$+ \frac{n-h}{n-r} \int_\Omega \frac{1}{|y-t|^{n-r}} \left[\int_\Omega \frac{\varphi(x,y)(x-y)}{|x-y|^{n+2-h}} d\sigma_x\right] d\sigma_y$$

$$= \int_\Omega \left[\int_\Omega \frac{\bar{\partial}_y \varphi(x,y)}{(n-r)|y-t|^{n-r}|x-y|^{n-h}} d\sigma_y\right] d\sigma_x$$

$$+ \frac{1}{n-r} \int_\Omega \frac{1}{|y-t|^{n-r}} \left[\int_\Omega \frac{\bar{\partial}_x \varphi(x,y)}{|x-y|^{n-h}} d\sigma_x\right] d\sigma_y$$

$$= \int_\Omega \left[\int_\Omega \frac{\bar{\partial}_y \varphi(x,y)}{(n-r)|y-t|^{n-r}|x-y|^{n-h}} d\sigma_y\right] d\sigma_x + \frac{1}{n-r}\int_\Omega \left[\int_\Omega \frac{\bar{\partial}_x \varphi(x,y)}{|y-t|^{n-r}|x-y|^{n-h}} d\sigma_y\right] d\sigma_x$$

$$= \frac{1}{n-r}\int_\Omega \left[\int_\Omega \frac{(\bar{\partial}_y + \bar{\partial}_x)\varphi(x,y)}{|y-t|^{n-r}|x-y|^{n-h}} d\sigma_y\right] d\sigma_x,$$

再利用定义 3.1, 引理 1.1, 引理 1.2, 和上式可得

$$\int_\Omega \left[\int_\Omega \frac{(y-t)\varphi(x,y)}{|y-t|^{n+2-r}|x-y|^{n-h}} d\sigma_y\right] d\sigma_x$$

$$= \int_\Omega \left[(\int_{\Omega^t} + \int_{\Omega^x}) \frac{(y-t)\varphi(x,y)d\sigma_y}{|y-t|^{n+2-r}|x-y|^{n-h}}\right] d\sigma_x$$

$$= \int_\Omega \left[\int_{\Omega^t} \frac{(y-t)\frac{\varphi(x,y)}{|x-y|^{n-h}}}{|y-t|^{n+2-r}} d\sigma_y\right] d\sigma_x + \int_\Omega \left[\int_{\Omega^x} \frac{(y-t)\varphi(x,y)d\sigma_y}{|y-t|^{n+2-r}|x-y|^{n-h}}\right] d\sigma_x$$

$$= \int_\Omega \left[\int_{\Omega^t} \frac{1}{(n-r)|y-t|^{n-r}} \bar{\partial}_y (\frac{\varphi(x,y)}{|x-y|^{n-h}}) d\sigma_y\right] d\sigma_x + \int_\Omega \left[\int_{\Omega^x} \frac{(y-t)\varphi(x,y)d\sigma_y}{|y-t|^{n+2-r}|x-y|^{n-h}}\right] d\sigma_x$$

$$= \int_\Omega \left[\int_{\Omega^t} \frac{\bar{\partial}_y \varphi(x,y)d\sigma_y}{(n-r)|y-t|^{n-r}|x-y|^{n-h}}\right] d\sigma_x + \int_\Omega \left[\int_{\Omega^t} \frac{\varphi(x,y)(n-h)(x-y)}{(n-r)|y-t|^{n-r}|x-y|^{n+2-h}} d\sigma_y\right] d\sigma_x$$

$$+ \int_\Omega \left[\int_{\Omega^x} \frac{1}{|x-y|^{n-h}(r-n)} \bar{\partial}_y (\frac{1}{|y-t|^{n-r}}) \varphi(x,y) d\sigma_y\right] d\sigma_x$$

$$+ \int_\Omega \left[\int_{\Omega^x} \frac{\bar{\partial}_y \varphi(x,y)d\sigma_y}{(n-r)|y-t|^{n-r}|x-y|^{n-h}}\right] d\sigma_x$$

$$- \int_\Omega \left[\int_{\Omega^x} \frac{\bar{\partial}_y \varphi(x,y)d\sigma_y}{(n-r)|y-t|^{n-r}|x-y|^{n-h}}\right] d\sigma_x$$

$$= \frac{1}{n-r}\int_{\Omega}\left[\int_{\Omega}\frac{\bar{\partial}_y\varphi(x,y)d\sigma_y}{|y-t|^{n-r}|x-y|^{n-h}}\right]d\sigma_x + \frac{1}{n-r}\int_{\Omega}\left[\int_{\Omega}\frac{\bar{\partial}_x\varphi(x,y)d\sigma_y}{|y-t|^{n-r}|x-y|^{n-h}}\right]d\sigma_x$$

$$-\frac{1}{n-r}\int_{\Omega}\left[\int_{\Omega^t}\frac{\bar{\partial}_x\varphi(x,y)d\sigma_y}{|y-t|^{n-r}|x-y|^{n-h}}\right]d\sigma_x + \frac{1}{n-r}\int_{\Omega}\left[\int_{\Omega^t}\frac{\varphi(x,y)(n-h)(x-y)}{|y-t|^{n-r}|x-y|^{n+2-h}}d\sigma_y\right]d\sigma_x$$

$$+\int_{\Omega}\left[\int_{\Omega^x}\frac{1}{(r-n)|x-y|^{n-h}}\bar{\partial}_y(\frac{1}{|y-t|^{n-r}})\varphi(x,y)d\sigma_y\right]d\sigma_x$$

$$-\int_{\Omega}\left[\int_{\Omega^x}\frac{\bar{\partial}_x\varphi(x,y)d\sigma_y}{(n-r)|y-t|^{n-r}|x-y|^{n-h}}\right]d\sigma_x$$

$$-\int_{\Omega}\left[\int_{\Omega^x}\frac{\bar{\partial}_y\varphi(x,y)d\sigma_y}{(n-r)|y-t|^{n-r}|x-y|^{n-h}}\right]d\sigma_x$$

$$= \frac{1}{n-r}\int_{\Omega}\left[\int_{\Omega}\frac{(\bar{\partial}_y+\bar{\partial}_x)\varphi(x,y)d\sigma_y}{|y-t|^{n-r}|x-y|^{n-h}}\right]d\sigma_x - \int_{\Omega}\left[\int_{\Omega^t}\frac{\bar{\partial}_x\varphi(x,y)d\sigma_y}{(n-r)|y-t|^{n-r}|x-y|^{n-h}}\right]d\sigma_x$$

$$-\int_{\Omega}\left[\int_{\Omega^t}\frac{\varphi(x,y)}{(n-r)|y-t|^{n-r}}\bar{\partial}_x\frac{1}{|x-y|^{n-h}}d\sigma_y\right]d\sigma_x$$

$$-\int_{\Omega}\left[\int_{\Omega^x}\frac{1}{(n-r)|x-y|^{n-h}}\bar{\partial}_y(\frac{1}{|y-t|^{n-r}})\varphi(x,y)d\sigma_y\right]d\sigma_x$$

$$-\int_{\Omega}\left[\int_{\Omega^x}\frac{(\bar{\partial}_y+\bar{\partial}_x)\varphi(x,y)d\sigma_y}{(n-r)|y-t|^{n-r}|x-y|^{n-h}}\right]d\sigma_x$$

$$= \frac{1}{n-r}\int_{\Omega}\left[\int_{\Omega}\frac{(\bar{\partial}_y+\bar{\partial}_x)\varphi(x,y)d\sigma_y}{|y-t|^{n-r}|x-y|^{n-h}}\right]d\sigma_x - \int_{\Omega}\left[\int_{\Omega^t}\frac{1}{(n-r)|y-t|^{n-r}}\bar{\partial}_x(\frac{\varphi(x,y)}{|x-y|^{n-h}})d\sigma_y\right]d\sigma_x$$

$$-\int_{\Omega}\left[\int_{\Omega^x}\frac{1}{(n-r)|x-y|^{n-h}}\bar{\partial}_y(\frac{1}{|y-t|^{n-r}})\varphi(x,y)d\sigma_y\right]d\sigma_x$$

$$-\int_{\Omega}\left[\int_{\Omega^x}\frac{(\bar{\partial}_y+\bar{\partial}_x)\varphi(x,y)d\sigma_y}{(n-r)|y-t|^{n-r}|x-y|^{n-h}}\right]d\sigma_x$$

$$= \int_\Omega \left[\frac{y-t}{|y-t|^{n+2-r}} \int_\Omega \frac{\varphi(x,y)d\sigma_x}{|x-y|^{n-h}} \right] d\sigma_y - \int_\Omega \left[\int_{\Omega^t} \frac{1}{(n-r)|y-t|^{n-r}} \bar{\partial}_x (\frac{\varphi(x,y)}{|x-y|^{n-h}}) d\sigma_y \right] d\sigma_x$$

$$- \int_\Omega \left[\int_{\Omega^x} \frac{1}{(n-r)|x-y|^{n-h}} \bar{\partial}_y (\frac{1}{|y-t|^{n-r}}) \varphi(x,y) d\sigma_y \right] d\sigma_x$$

$$- \int_\Omega \left[\int_{\Omega^x} \frac{(\bar{\partial}_y + \bar{\partial}_x)\varphi(x,y)d\sigma_y}{(n-r)|y-t|^{n-r}|x-y|^{n-h}} \right] d\sigma_x,$$

所以

$$\int_\Omega \left[\frac{y-t}{|y-t|^{n+2-r}} \int_\Omega \frac{\varphi(x,y)d\sigma_x}{|x-y|^{n-h}} \right] d\sigma_y$$

$$= \int_\Omega \left[\int_\Omega \frac{(y-t)\varphi(x,y)}{|y-t|^{n+2-r}|x-y|^{n-h}} d\sigma_y \right] d\sigma_x + \int_\Omega \left[\int_{\Omega^t} \frac{1}{(n-r)|y-t|^{n-r}} \bar{\partial}_x (\frac{\varphi(x,y)}{|x-y|^{n-h}}) d\sigma_y \right] d\sigma_x$$

$$+ \int_\Omega \left[\int_{\Omega^x} \frac{1}{(n-r)|x-y|^{n-h}} \bar{\partial}_y (\frac{1}{|y-t|^{n-1-r}}) \varphi(x,y) d\sigma_y \right] d\sigma_x$$

$$+ \int_\Omega \left[\int_{\Omega^x} \frac{(\bar{\partial}_y + \bar{\partial}_x)\varphi(x,y)d\sigma_y}{(n-r)|y-t|^{n-r}|x-y|^{n-h}} \right] d\sigma_x$$

$$= \int_\Omega \left[\int_\Omega \frac{(y-t)\varphi(x,y)}{|y-t|^{n+2-r}|x-y|^{n-h}} d\sigma_y \right] d\sigma_x + \int_\Omega \left[\int_{\Omega^t} \frac{1}{(n-r)|y-t|^{n-r}} \bar{\partial}_x (\frac{\varphi(x,y)}{|x-y|^{n-h}}) d\sigma_y \right] d\sigma_x$$

$$+ \int_\Omega \left[\int_{\Omega^x} \frac{1}{(n-r)|x-y|^{n-h}} \bar{\partial}_y (\frac{\varphi(x,y)}{|y-t|^{n-r}}) d\sigma_y \right] d\sigma_x$$

$$+ \int_\Omega \left[\int_{\Omega^x} \frac{\bar{\partial}_x \varphi(x,y)d\sigma_y}{(n-r)|y-t|^{n-r}|x-y|^{n-h}} \right] d\sigma_x$$

$$= \int_\Omega \left[\int_\Omega \frac{(y-t)\varphi(x,y)}{|y-t|^{n+2-r}|x-y|^{n-h}} d\sigma_y \right] d\sigma_x + \int_\Omega \left[\int_{\Omega^t} \frac{1}{(n-r)|y-t|^{n-r}} \bar{\partial}_x (\frac{\varphi(x,y)}{|x-y|^{n-h}}) d\sigma_y \right] d\sigma_x$$

$$+\int_{\Omega}\left[\int_{\Omega^x}\frac{1}{(n-r)|x-y|^{n-h}}\bar{\partial}_y(\frac{\varphi(x,y)}{|y-t|^{n-r}})d\sigma_y\right]d\sigma_x$$

$$+\int_{\Omega}\left[\int_{\Omega^x}\frac{1}{(n-r)|x-y|^{n-1-h}}\bar{\partial}_x(\frac{\varphi(x,y)}{|y-t|^{n-r}})d\sigma_y\right]d\sigma_x$$

$$=\int_{\Omega}\left[\int_{\Omega}\frac{(y-t)\varphi(x,y)}{|y-t|^{n+2-r}|x-y|^{n-h}}d\sigma_y\right]d\sigma_x+\int_{\Omega}\left[\int_{\Omega^t}\frac{1}{(n-r)|y-t|^{n-r}}\bar{\partial}_x(\frac{\varphi(x,y)}{|x-y|^{n-h}})d\sigma_y\right]d\sigma_x$$

$$+\int_{\Omega}\left[\int_{\Omega^x}\frac{(\bar{\partial}_x+\bar{\partial}_y)}{(n-r)|x-y|^{n-h}}(\frac{\varphi(x,y)}{|y-t|^{n-r}})d\sigma_y\right]d\sigma_x,$$

(3.13) 式得证. 类似地可证 (3.14) 式. **证毕**.

定理 3.8 设 $\varphi(x,y)\in H^{(4,4)}(\beta_1,\beta_2)$, $0<\beta_i<1$, r, $h>\beta_i$, $i=1,2$, $n>0$, $0\le r<2$, $n>h+r$, $\lambda_5=\frac{n-h}{2(1-r)(n-r)}$, $\lambda_6=\frac{1}{(1-r)(n-r)}$, 则

$$\int_{\Omega}\frac{1}{|y-t|^{n+2-r}}\int_{\Omega}\frac{\varphi(x,y)d\sigma_x}{|x-y|^{n-h}}d\sigma_y$$

$$=\int_{\Omega}\left[\int_{\Omega}\frac{\varphi(x,y)}{|y-t|^{n+2-r}|x-y|^{n-h}}d\sigma_y\right]d\sigma_x$$

$$-\lambda_5\int_{\Omega}\left[\int_{\Omega^t}\frac{\bar{\partial}_y\varphi(x,y)(\bar{x}-\bar{y})+\partial_y\varphi(x,y)(x-y)}{|x-y|^{n+2-h}|y-t|^{n-r}}d\sigma_y\right]d\sigma_x$$

$$-\lambda_5\int_{\Omega}\left[\int_{\Omega^t}\frac{\partial_y(\frac{\varphi(x,y)(x-y)}{|x-y|^{n+2-h}})+\bar{\partial}_y(\frac{\varphi(x,y)(\bar{x}-\bar{y})}{|x-y|^{n+2-h}})}{|y-t|^{n-r}}d\sigma_y\right]d\sigma_x-\int_{\Omega}\left[\int_{\Omega^x}\frac{\varphi(x,y)}{|y-t|^{n+2-r}|x-y|^{n-h}}d\sigma_y\right]d\sigma_x$$

$$+\lambda_6\int_{\Omega}\left[\int_{\Omega^x}\frac{\Delta_x\varphi(x,y)}{|x-y|^{n-h}|y-t|^{n-r}}d\sigma_y\right]d\sigma_x+\lambda_6\int_{\Omega}\left[\int_{\Omega}\frac{(\Delta_x+\partial_x\bar{\partial}_y+\partial_y\bar{\partial}_x)\varphi(x,y)}{|x-y|^{n-h}|y-t|^{n-r}}d\sigma_y\right]d\sigma_x. \tag{3.15}$$

证明 在题设条件下，利用引理 1.1, 引理 1.2, 和引理 3.1 可得

$$\int_\Omega \frac{1}{|y-t|^{n+2-r}} \int_\Omega \frac{\varphi(x,y)d\sigma_x}{|x-y|^{n-h}} d\sigma_y$$

$$= \frac{1}{2(1-r)} \int_\Omega \frac{\bar{y}-\bar{t}}{|y-t|^{n+2-r}} \left[\int_\Omega \bar{\partial}_y \frac{\varphi(x,y)}{|x-y|^{n-h}} d\sigma_x\right] d\sigma_y$$

$$+\frac{1}{2(1-r)} \int_\Omega \frac{y-t}{|y-t|^{n+2-r}} \left[\int_\Omega \partial_y \frac{\varphi(x,y)}{|x-y|^{n-h}} d\sigma_x\right] d\sigma_y$$

$$= \frac{1}{2(1-r)} \int_\Omega \frac{\bar{y}-\bar{t}}{|y-t|^{n+2-r}} \left[\int_\Omega \frac{\bar{\partial}_y\varphi(x,y)}{|x-y|^{n-h}} d\sigma_x + \int_\Omega \frac{\varphi(x,y)(x-y)(n-h)}{|x-y|^{n+2-h}} d\sigma_x\right] d\sigma_y$$

$$+\frac{1}{2(1-r)} \int_\Omega \frac{y-t}{|y-t|^{n+2-r}} \left[\int_\Omega \frac{\partial_y\varphi(x,y)}{|x-y|^{n-h}} d\sigma_x + \int_\Omega \frac{\varphi(x,y)(\bar{x}-\bar{y})(n-h)}{|x-y|^{n+2-h}} d\sigma_x\right] d\sigma_y$$

$$= \frac{1}{2(1-r)} \int_\Omega \frac{\bar{y}-\bar{t}}{|y-t|^{n+2-r}} \left[\int_\Omega \frac{\bar{\partial}_y\varphi(x,y)}{|x-y|^{n-h}} d\sigma_x\right] d\sigma_y$$

$$+\frac{1}{2(1-r)} \int_\Omega \frac{\bar{y}-\bar{t}}{|y-t|^{n+2-r}} \left[\int_\Omega \frac{\varphi(x,y)(x-y)(n-h)}{|x-y|^{n+2-h}} d\sigma_x\right] d\sigma_y$$

$$+\frac{1}{2(1-r)} \int_\Omega \frac{y-t}{|y-t|^{n+2-r}} \left[\int_\Omega \frac{\partial_y\varphi(x,y)}{|x-y|^{n-h}} d\sigma_x\right] d\sigma_y$$

$$+\frac{1}{2(1-r)} \int_\Omega \frac{y-t}{|y-t|^{n+2-r}} \left[\int_\Omega \frac{\varphi(x,y)(\bar{x}-\bar{y})(n-h)}{|x-y|^{n+2-h}} d\sigma_x\right] d\sigma_y$$

$$= \frac{1}{2(1-r)} \int_\Omega \frac{1}{(n-r)|y-t|^{n-r}} \left[\int_\Omega \partial_y\left(\frac{\bar{\partial}_y\varphi(x,y)}{|x-y|^{n-h}}\right) d\sigma_x\right] d\sigma_y$$

$$+\frac{1}{2(1-r)} \int_\Omega \frac{\bar{y}-\bar{t}}{|y-t|^{n+2-r}} \left[\int_\Omega \frac{\bar{\partial}_x\varphi(x,y)}{|x-y|^{n-h}} d\sigma_x\right] d\sigma_y$$

$$+\frac{1}{2(1-r)}\int_{\Omega}\frac{1}{(n-r)|y-t|^{n-r}}\left[\int_{\Omega}\bar{\partial}_y(\frac{\partial_y\varphi(x,y)}{|x-y|^{n-h}})d\sigma_x\right]d\sigma_y$$

$$+\frac{1}{2(1-r)}\int_{\Omega}\frac{y-t}{|y-t|^{n+2-r}}\left[\int_{\Omega}\frac{\partial_x\varphi(x,y)}{|x-y|^{n-h}}d\sigma_x\right]d\sigma_y$$

$$=\frac{1}{2(1-r)}\int_{\Omega}\frac{1}{(n-r)|y-t|^{n-r}}\int_{\Omega}\frac{\Delta_y\varphi(x,y)}{|x-y|^{n-h}}d\sigma_x d\sigma_y$$

$$+\frac{1}{2(1-r)}\int_{\Omega}\int_{\Omega}\frac{\bar{\partial}_y\varphi(x,y)(\bar{x}-\bar{y})(n-h)}{(n-r)|y-t|^{n-r}|x-y|^{n+2-h}}d\sigma_x d\sigma_y$$

$$+\frac{1}{2(1-r)}\int_{\Omega}\frac{1}{(n-r)|y-t|^{n-r}}\int_{\Omega}\partial_y\frac{\bar{\partial}_x\varphi(x,y)}{|x-y|^{n-h}}d\sigma_x d\sigma_y$$

$$+\frac{1}{2(1-r)}\int_{\Omega}\frac{1}{(n-r)|y-t|^{n-r}}\int_{\Omega}\frac{\Delta_y\varphi(x,y)}{|x-y|^{n-h}}d\sigma_x d\sigma_y$$

$$+\frac{1}{2(1-r)}\int_{\Omega}\int_{\Omega}\frac{\partial_y\varphi(x,y)(x-y)(n-h)}{|x-y|^{n+2-h}}d\sigma_x d\sigma_y$$

$$+\frac{1}{2(1-r)}\int_{\Omega}\frac{1}{(n-r)|y-t|^{n-r}}\left[\int_{\Omega}\bar{\partial}_y(\frac{\partial_x\varphi(x,y)}{(n-r)|y-t|^{n-r}|x-y|^{n-h}})d\sigma_x\right]d\sigma_y$$

$$=\frac{1}{2(1-r)(n-r)}\int_{\Omega}\frac{1}{|y-t|^{n-r}}\left[\int_{\Omega}\frac{\Delta_y\varphi(x,y)}{|x-y|^{n-h}}d\sigma_x\right]d\sigma_y$$

$$+\frac{1}{2(1-r)(n-r)}\int_{\Omega}\frac{1}{|y-t|^{n-r}}\left[\int_{\Omega}\frac{\bar{\partial}_y\varphi(x,y)(\bar{x}-\bar{y})(n-h)}{|x-y|^{n+2-h}}d\sigma_x\right]d\sigma_y$$

$$+\frac{1}{2(1-r)(n-r)}\int_{\Omega}\frac{1}{|y-t|^{n-r}}\int_{\Omega}\frac{\partial_y\bar{\partial}_x\varphi(x,y)}{|x-y|^{n-h}}d\sigma_x d\sigma_y$$

$$+\frac{1}{2(1-r)(n-r)}\int_\Omega \frac{1}{|y-t|^{n-r}}\int_\Omega \frac{\bar{\partial}_x\varphi(x,y)(\bar{x}-\bar{y})(n-h)}{|x-y|^{n+2-h}}d\sigma_x d\sigma_y$$

$$+\frac{1}{2(1-r)(n-r)}\int_\Omega \frac{1}{|y-t|^{n-r}}\left[\int_\Omega \frac{\Delta_y\varphi(x,y)}{|x-y|^{n-h}}d\sigma_x\right]d\sigma_y$$

$$+\frac{1}{2(1-r)(n-r)}\int_\Omega \frac{1}{|y-t|^{n-r}}\left[\int_\Omega \frac{\partial_y\varphi(x,y)(x-y)(n-h)}{|x-y|^{n+2-h}}d\sigma_x\right]d\sigma_y$$

$$+\frac{1}{2(1-r)(n-r)}\int_\Omega \frac{1}{|y-t|^{n-r}}\int_\Omega \frac{\bar{\partial}_y\partial_x\varphi(x,y)}{|x-y|^{n-h}}d\sigma_x d\sigma_y$$

$$+\frac{1}{2(1-r)(n-r)}\int_\Omega \frac{1}{|y-t|^{n-r}}\int_\Omega \frac{\partial_x\varphi(x,y)(x-y)(n-h)}{|x-y|^{n+2-h}}d\sigma_x d\sigma_y$$

$$= \lambda_6\int_\Omega\left[\int_\Omega \frac{\Delta_y\varphi(x,y)}{|y-t|^{n-r}|x-y|^{n-h}}d\sigma_y\right]d\sigma_x$$

$$+\frac{1}{2(1-r)(n-r)}\int_\Omega \frac{1}{|y-t|^{n-r}}\left[\int_\Omega \frac{\partial_x\bar{\partial}_y\varphi(x,y)}{|x-y|^{n-h}}d\sigma_x\right]d\sigma_y$$

$$+\frac{1}{2(1-r)(n-r)}\int_\Omega\left[\int_\Omega \frac{\partial_y\bar{\partial}_x\varphi(x,y)}{|y-t|^{n-r}|x-y|^{n-h}}d\sigma_y\right]d\sigma_x$$

$$+\frac{1}{2(1-r)(n-r)}\int_\Omega \frac{1}{|y-t|^{n-r}}\left[\int_\Omega \frac{\Delta_x\varphi(x,y)}{|x-y|^{n-h}}d\sigma_x\right]d\sigma_y$$

$$+\frac{1}{2(1-r)(n-r)}\int_\Omega \frac{1}{|y-t|^{n-r}}\left[\int_\Omega \frac{\bar{\partial}_x\partial_y\varphi(x,y)}{|x-y|^{n-h}}d\sigma_x\right]d\sigma_y$$

$$+\frac{1}{2(1-r)(n-r)}\int_\Omega\left[\int_\Omega \frac{\bar{\partial}_y\partial_x\varphi(x,y)}{|y-t|^{n-r}|x-y|^{n-h}}d\sigma_y\right]d\sigma_x$$

$$+\frac{1}{2(1-r)(n-r)}\int_\Omega \frac{1}{|y-t|^{n-r}}\left[\int_\Omega \frac{\Delta_x\varphi(x,y)}{|x-y|^{n-h}}d\sigma_x\right]d\sigma_y$$

$$
\begin{aligned}
= \ & \lambda_6 \int_\Omega \left[\int_\Omega \frac{\Delta_y \varphi(x,y)}{|y-t|^{n-r}|x-y|^{n-h}} d\sigma_y \right] d\sigma_x + \lambda_6 \int_\Omega \left[\int_\Omega \frac{\partial_x \bar{\partial}_y \varphi(x,y)}{|y-t|^{n-r}|x-y|^{n-h}} d\sigma_y \right] d\sigma_x \\
& + \lambda_6 \int_\Omega \left[\int_\Omega \frac{\partial_y \bar{\partial}_x \varphi(x,y)}{|y-t|^{n-r}|x-y|^{n-h}} d\sigma_y \right] d\sigma_x + \lambda_6 \int_\Omega \left[\int_\Omega \frac{\Delta_x \varphi(x,y)}{|y-t|^{n-r}|x-y|^{n-h}} d\sigma_y \right] d\sigma_x \\
= \ & \lambda_6 \int_\Omega \left[\int_\Omega \frac{(\Delta_y + \Delta_x)\varphi(x,y)}{|y-t|^{n-r}|x-y|^{n-h}} d\sigma_y \right] d\sigma_x + \lambda_6 \int_\Omega \left[\int_\Omega \frac{(\partial_x \bar{\partial}_y + \partial_y \bar{\partial}_x)\varphi(x,y)}{|y-t|^{n-r}|x-y|^{n-h}} d\sigma_y \right] d\sigma_x,
\end{aligned}
$$

再利用定义 3.1, 引理 1.1, 引理 1.2 和上式可得

$$
\begin{aligned}
& \int_\Omega \left[\int_\Omega \frac{\varphi(x,y)}{|y-t|^{n+2-r}|x-y|^{n-h}} d\sigma_y \right] d\sigma_x \\
= \ & \int_\Omega \left[\int_{\Omega^t} \frac{\varphi(x,y)}{|x-y|^{n-h}|y-t|^{n+2-r}} d\sigma_y \right] d\sigma_x + \int_\Omega \left[\int_{\Omega^x} \frac{\varphi(x,y)}{|y-t|^{n+2-r}|x-y|^{n-h}} d\sigma_y \right] d\sigma_x \\
= \ & \int_\Omega \left[\int_{\Omega^t} \frac{(\bar{y}-\bar{t})\bar{\partial}_y \left(\frac{\varphi(x,y)}{|x-y|^{n-h}}\right)}{2(1-r)|y-t|^{n+2-r}} d\sigma_y \right] d\sigma_x + \int_\Omega \left[\int_{\Omega^t} \frac{(y-t)\partial_y \left(\frac{\varphi(x,y)}{|x-y|^{n-h}}\right)}{2(1-r)|y-t|^{n+2-r}} d\sigma_y \right] d\sigma_x \\
& + \int_\Omega \left[\int_{\Omega^x} \frac{\varphi(x,y)}{|y-t|^{n+2-r}|x-y|^{n-h}} d\sigma_y \right] d\sigma_x \\
= \ & \frac{1}{2(1-r)} \int_\Omega \left[\int_{\Omega^t} \frac{(\bar{y}-\bar{t}) \frac{\bar{\partial}_y \varphi(x,y)}{|x-y|^{n-h}}}{|y-t|^{n+2-r}} d\sigma_y \right] d\sigma_x \\
& + \frac{1}{2(1-r)} \int_\Omega \left[\int_{\Omega^t} \frac{(\bar{y}-\bar{t}) \frac{\varphi(x,y)(x-y)(n-h)}{|x-y|^{n+2-h}}}{|y-t|^{n+2-r}} d\sigma_y \right] d\sigma_x \\
& + \frac{1}{2(1-r)} \int_\Omega \left[\int_{\Omega^t} \frac{(y-t) \frac{\partial_y \varphi(x,y)}{|x-y|^{n-h}}}{|y-t|^{n+2-r}} d\sigma_y \right] d\sigma_x
\end{aligned}
$$

$$+\frac{1}{2(1-r)}\int_\Omega\left[\int_{\Omega^t}\frac{(y-t)\frac{\varphi(x,y)(\bar{x}-\bar{y})(n-h)}{|x-y|^{n+2-h}}}{|y-t|^{n+2-r}}d\sigma_y\right]d\sigma_x$$

$$+\int_\Omega\left[\int_{\Omega^x}\frac{\varphi(x,y)}{|y-t|^{n+2-r}|x-y|^{n-h}}d\sigma_y\right]d\sigma_x$$

$$=\frac{1}{2(1-r)}\int_\Omega\left[\int_{\Omega^t}\frac{\partial_y(\frac{\bar{\partial}_y\varphi(x,y)}{|x-y|^{n-h}})}{(n-r)|y-t|^{n-r}}d\sigma_y\right]d\sigma_x$$

$$+\frac{1}{2(1-r)}\int_\Omega\left[\int_{\Omega^t}\frac{\partial_y(\frac{\varphi(x,y)(x-y)(n-h)}{|x-y|^{n+2-h}})}{(n-r)|y-t|^{n-r}}d\sigma_y\right]d\sigma_x$$

$$+\frac{1}{2(1-r)}\int_\Omega\left[\int_{\Omega^t}\frac{\bar{\partial}_y(\frac{\partial_y\varphi(x,y)}{|x-y|^{n-h}})}{(n-r)|y-t|^{n-r}}d\sigma_y\right]d\sigma_x$$

$$+\frac{1}{2(1-r)}\int_\Omega\left[\int_{\Omega^t}\frac{\bar{\partial}_y(\frac{\varphi(x,y)(\bar{x}-\bar{y})(n-h)}{|x-y|^{n+2-h}})}{(n-r)|y-t|^{n-r}}d\sigma_y\right]d\sigma_x$$

$$+\frac{1}{2(1-r)}\int_\Omega\left[\int_{\Omega^x}\frac{\varphi(x,y)}{|y-t|^{n+2-r}|x-y|^{n-h}}d\sigma_y\right]d\sigma_x$$

$$=\frac{1}{2(1-r)(n-r)}\int_\Omega\left[\int_{\Omega^t}\frac{\Delta_y\varphi(x,y)}{|y-t|^{n-r}|x-y|^{n-h}}+\int_{\Omega^t}\frac{\bar{\partial}_y\varphi(x,y)(\bar{x}-\bar{y})(n-h)}{|x-y|^{n+2-h}|y-t|^{n-r}}d\sigma_y\right]d\sigma_x$$

$$+\lambda_5\int_\Omega\left[\int_{\Omega^t}\frac{\partial_y(\frac{\varphi(x,y)(x-y)}{|x-y|^{n+2-h}})}{|y-t|^{n-r}}d\sigma_y\right]d\sigma_x+\lambda_5\int_\Omega\left[\int_{\Omega^t}\frac{\bar{\partial}_y(\frac{\varphi(x,y)(\bar{x}-\bar{y})}{|x-y|^{n+2-h}})}{|y-t|^{n-r}}d\sigma_y\right]d\sigma_x$$

$$+\frac{1}{2(1-r)(n-r)}\int_\Omega\left[\int_{\Omega^t}\frac{\Delta_y\varphi(x,y)}{|y-t|^{n-r}|x-y|^{n-h}}d\sigma_y\right]d\sigma_x$$

$$
\begin{aligned}
&+\lambda_5 \frac{1}{2(1-r)(n-r)} \int_\Omega \left[\int_{\Omega^t} \frac{\partial_y \varphi(x,y)(x-y)(n-h)}{|x-y|^{n+2-h}|y-t|^{n-r}} d\sigma_y \right] d\sigma_x \\
&+\int_\Omega \left[\int_{\Omega^x} \frac{\varphi(x,y)}{|y-t|^{n+2-r}|x-y|^{n-h}} d\sigma_y \right] d\sigma_x \\
= \; & \lambda_6 \int_\Omega \left[\int_{\Omega^t} \frac{\Delta_y \varphi(x,y)}{|y-t|^{n-r}|x-y|^{n-h}} d\sigma_y \right] d\sigma_x + \lambda_5 \int_\Omega \left[\int_{\Omega^t} \frac{\bar\partial_y \varphi(x,y)(\bar x - \bar y)}{|x-y|^{n+2-h}|y-t|^{n-r}} d\sigma_y \right] d\sigma_x \\
&+\lambda_5 \int_\Omega \left[\int_{\Omega^t} \frac{\partial_y \varphi(x,y)(x-y)}{|x-y|^{n+2-h}|y-t|^{n-r}} d\sigma_y \right] d\sigma_x + \lambda_5 \int_\Omega \left[\int_{\Omega^t} \frac{\partial_y \left(\frac{\varphi(x,y)(x-y)}{|x-y|^{n+2-h}}\right)}{|y-t|^{n-r}} d\sigma_y \right] d\sigma_x \\
&+\lambda_5 \int_\Omega \left[\int_{\Omega^t} \frac{\bar\partial_y \left(\frac{\varphi(x,y)(\bar x-\bar y)}{|x-y|^{n+2-h}}\right)}{|y-t|^{n-r}} d\sigma_y \right] d\sigma_x + \int_\Omega \left[\int_{\Omega^x} \frac{\varphi(x,y)}{|y-t|^{n+2-r}|x-y|^{n-h}} d\sigma_y \right] d\sigma_x \\
= \; & \lambda_6 \int_\Omega \left[\int_{\Omega^t} \frac{\Delta_y \varphi(x,y)}{|y-t|^{n-r}|x-y|^{n-h}} d\sigma_y \right] d\sigma_x + \lambda_6 \int_\Omega \left[\int_{\Omega^x} \frac{\Delta_y \varphi(x,y)}{|y-t|^{n-r}|x-y|^{n-h}} d\sigma_y \right] d\sigma_x \\
&+\lambda_5 \int_\Omega \left[\int_{\Omega^t} \frac{\bar\partial_y \varphi(x,y)(\bar x - \bar y)}{|x-y|^{n+2-h}|y-t|^{n-r}} d\sigma_y \right] d\sigma_x + \lambda_5 \int_\Omega \left[\int_{\Omega^t} \frac{\partial_y \varphi(x,y)(x-y)}{|x-y|^{n+2-h}|y-t|^{n-r}} d\sigma_y \right] d\sigma_x \\
&+\lambda_5 \int_\Omega \left[\int_{\Omega^t} \frac{\partial_y \left(\frac{\varphi(x,y)(x-y)}{|x-y|^{n+2-h}}\right)}{|y-t|^{n-r}} d\sigma_y \right] d\sigma_x + \lambda_5 \int_\Omega \left[\int_{\Omega^t} \frac{\bar\partial_y \left(\frac{\varphi(x,y)(\bar x-\bar y)}{|x-y|^{n+2-h}}\right)}{|y-t|^{n-r}} d\sigma_y \right] d\sigma_x \\
&+\int_\Omega \left[\int_{\Omega^x} \frac{\varphi(x,y)}{|x-y|^{n-h}|y-t|^{n+2-r}} d\sigma_y \right] d\sigma_x - \lambda_6 \int_\Omega \left[\int_{\Omega^x} \frac{\Delta_y \varphi(x,y)}{|y-t|^{n-r}|x-y|^{n-h}} d\sigma_y \right] d\sigma_x \\
= \; & \lambda_6 \int_\Omega \left[\int_{\Omega} \frac{\Delta_y \varphi(x,y)}{|y-t|^{n-r}|x-y|^{n-h}} d\sigma_y \right] d\sigma_x + \lambda_5 \int_\Omega \left[\int_{\Omega^t} \frac{\bar\partial_y \varphi(x,y)(\bar x - \bar y)}{|x-y|^{n+2-h}|y-t|^{n-r}} d\sigma_y \right] d\sigma_x
\end{aligned}
$$

$$+\lambda_5\int_{\Omega}\left[\int_{\Omega^t}\frac{\partial_y\varphi(x,y)(x-y)}{|x-y|^{n+2-h}|y-t|^{n-r}}d\sigma_y\right]d\sigma_x+\lambda_5\int_{\Omega}\left[\int_{\Omega^t}\frac{\partial_y\left(\frac{\varphi(x,y)(x-y)}{|x-y|^{n+2-h}}\right)}{|y-t|^{n-r}}d\sigma_y\right]d\sigma_x$$

$$+\lambda_5\int_{\Omega}\left[\int_{\Omega^t}\frac{\bar\partial_y\left(\frac{\varphi(x,y)(\bar x-\bar y)}{|x-y|^{n+2-h}}\right)}{|y-t|^{n-r}}d\sigma_y\right]d\sigma_x+\int_{\Omega}\left[\int_{\Omega^x}\frac{\varphi(x,y)}{|x-y|^{n-h}|y-t|^{n+2-r}}d\sigma_y\right]d\sigma_x$$

$$-\lambda_6\int_{\Omega}\left[\int_{\Omega^x}\frac{\Delta_y\varphi(x,y)}{|y-t|^{n-r}|x-y|^{n-h}}d\sigma_y\right]d\sigma_x+\lambda_6\int_{\Omega}\left[\int_{\Omega}\frac{\Delta_x\varphi(x,y)}{|y-t|^{n-r}|x-y|^{n-h}}d\sigma_y\right]d\sigma_x$$

$$-\lambda_6\int_{\Omega}\left[\int_{\Omega}\frac{\Delta_x\varphi(x,y)}{|y-t|^{n-r}|x-y|^{n-h}}d\sigma_y\right]d\sigma_x$$

$$=\lambda_6\int_{\Omega}\left[\int_{\Omega}\frac{(\Delta_y+\Delta_x)\varphi(x,y)}{|y-t|^{n-r}|x-y|^{n-h}}d\sigma_y\right]d\sigma_x+\lambda_5\int_{\Omega}\left[\int_{\Omega^t}\frac{\bar\partial_y\varphi(x,y)(\bar x-\bar y)}{|x-y|^{n+2-h}|y-t|^{n-r}}d\sigma_y\right]d\sigma_x$$

$$+\lambda_5\int_{\Omega}\left[\int_{\Omega^t}\frac{\partial_y\varphi(x,y)(x-y)}{|x-y|^{n+2-h}|y-t|^{n-r}}d\sigma_y\right]d\sigma_x+\lambda_5\int_{\Omega}\left[\int_{\Omega^t}\frac{\partial_y\left(\frac{\varphi(x,y)(x-y)}{|x-y|^{n+2-h}}\right)}{|y-t|^{n-r}}d\sigma_y\right]d\sigma_x$$

$$+\lambda_5\int_{\Omega}\left[\int_{\Omega^t}\frac{\bar\partial_y\left(\frac{\varphi(x,y)(\bar x-\bar y)}{|x-y|^{n+2-h}}\right)}{|y-t|^{n-r}}d\sigma_y\right]d\sigma_x+\int_{\Omega}\left[\int_{\Omega^x}\frac{\varphi(x,y)}{|x-y|^{n-h}|y-t|^{n+2-r}}d\sigma_y\right]d\sigma_x$$

$$-\lambda_6\int_{\Omega}\left[\int_{\Omega^x}\frac{\Delta_y\varphi(x,y)}{|y-t|^{n-r}|x-y|^{n-h}}d\sigma_y\right]d\sigma_x-\lambda_6\int_{\Omega}\left[\int_{\Omega^x}\frac{\Delta_x\varphi(x,y)}{|y-t|^{n-r}|x-y|^{n-h}}d\sigma_y\right]d\sigma_x$$

$$-\lambda_6\int_{\Omega}\left[\int_{\Omega^t}\frac{\Delta_x\varphi(x,y)}{|y-t|^{n-r}|x-y|^{n-h}}d\sigma_y\right]d\sigma_x$$

$$=\lambda_6\int_{\Omega}\left[\int_{\Omega}\frac{(\Delta_y+\Delta_x)\varphi(x,y)}{|y-t|^{n-r}|x-y|^{n-h}}d\sigma_y\right]d\sigma_x$$

$$+\lambda_5\int_\Omega\left[\int_{\Omega^t}\frac{\bar\partial_y\varphi(x,y)(\bar x-\bar y)}{|x-y|^{n+2-h}|y-t|^{n-r}}d\sigma_y\right]d\sigma_x$$

$$+\lambda_5\int_\Omega\left[\int_{\Omega^t}\frac{\partial_y\varphi(x,y)(x-y)}{|x-y|^{n+2-h}|y-t|^{n-r}}d\sigma_y\right]d\sigma_x+\lambda_5\int_\Omega\left[\int_{\Omega^t}\frac{\partial_y(\frac{\varphi(x,y)(x-y)}{|x-y|^{n+2-h}})}{|y-t|^{n-r}}d\sigma_y\right]d\sigma_x$$

$$+\lambda_5\int_\Omega\left[\int_{\Omega^t}\frac{\bar\partial_y(\frac{\varphi(x,y)(\bar x-\bar y)}{|x-y|^{n+2-h}})}{|y-t|^{n-r}}d\sigma_y\right]d\sigma_x+\int_\Omega\left[\int_{\Omega^x}\frac{\varphi(x,y)}{|x-y|^{n-h}|y-t|^{n+2-r}}d\sigma_y\right]d\sigma_x$$

$$-\lambda_6\int_\Omega\left[\int_{\Omega^x}\frac{(\Delta_x+\Delta_y)\varphi(x,y)}{|y-t|^{n-r}|x-y|^{n-h}}d\sigma_y\right]d\sigma_x-\lambda_6\int_\Omega\left[\int_{\Omega^t}\frac{\Delta_x\varphi(x,y)}{|y-t|^{n-r}|x-y|^{n-h}}d\sigma_y\right]d\sigma_x,$$

所以

$$\int_\Omega\frac{1}{|y-t|^{n+2-r}}(\int_\Omega\frac{\varphi(x,y)}{|x-y|^{n-h}}d\sigma_x)d\sigma_y$$

$$=\int_\Omega\left[\int_\Omega\frac{\varphi(x,y)}{|y-t|^{n+2-r}|x-y|^{n-h}}d\sigma_y\right]d\sigma_x-\lambda_5\int_\Omega\left[\int_{\Omega^t}\frac{\bar\partial_y\varphi(x,y)(\bar x-\bar y)}{|x-y|^{n+2-h}|y-t|^{n-r}}d\sigma_y\right]d\sigma_x$$

$$-\lambda_5\int_\Omega\left[\int_{\Omega^t}\frac{\partial_y\varphi(x,y)(x-y)}{|x-y|^{n+2-h}|y-t|^{n-r}}d\sigma_y\right]d\sigma_x-\lambda_5\int_\Omega\left[\int_{\Omega^t}\frac{\partial_y(\frac{\varphi(x,y)(x-y)}{|x-y|^{n+2-h}})}{|y-t|^{n-r}}d\sigma_y\right]d\sigma_x$$

$$-\lambda_5\int_\Omega\left[\int_{\Omega^t}\frac{\bar\partial_y(\frac{\varphi(x,y)(\bar x-\bar y)}{|x-y|^{n+2-h}})}{|y-t|^{n-r}}d\sigma_y\right]d\sigma_x-\int_\Omega\left[\int_{\Omega^x}\frac{\varphi(x,y)}{|y-t|^{n+2-r}|x-y|^{n-h}}d\sigma_y\right]d\sigma_x$$

$$+\lambda_6\int_\Omega\left[\int_{\Omega^x}\frac{(\Delta_x+\Delta_y)\varphi(x,y)}{|x-y|^{n-h}|y-t|^{n-r}}d\sigma_y\right]d\sigma_x+\lambda_6\int_\Omega\left[\int_{\Omega^t}\frac{\Delta_x\varphi(x,y)}{|x-y|^{n-h}|y-t|^{n-r}}d\sigma_y\right]d\sigma_x$$

$$+\lambda_6\int_\Omega\left[\int_\Omega\frac{(\partial_x\bar\partial_y+\partial_y\bar\partial_x)\varphi(x,y)}{|x-y|^{n-h}|y-t|^{n-r}}d\sigma_y\right]d\sigma_x$$

$$= \int_\Omega \left[\int_\Omega \frac{\varphi(x,y)}{|y-t|^{n+2-r}|x-y|^{n-h}} d\sigma_y\right] d\sigma_x - \lambda_5 \int_\Omega \left[\int_{\Omega^t} \frac{\bar{\partial}_y \varphi(x,y)(\bar{x}-\bar{y})}{|x-y|^{n+2-h}|y-t|^{n-r}} d\sigma_y\right] d\sigma_x$$

$$-\lambda_5 \int_\Omega \left[\int_{\Omega^t} \frac{\partial_y \varphi(x,y)(x-y)}{|x-y|^{n+2-h}|y-t|^{n-r}} d\sigma_y\right] d\sigma_x - \lambda_5 \int_\Omega \left[\int_{\Omega^t} \frac{\partial_y \left(\frac{\varphi(x,y)(x-y)}{|x-y|^{n+2-h}}\right)}{|y-t|^{n-r}} d\sigma_y\right] d\sigma_x$$

$$-\lambda_5 \int_\Omega \left[\int_{\Omega^t} \frac{\bar{\partial}_y \left(\frac{\varphi(x,y)(\bar{x}-\bar{y})}{|x-y|^{n+2-h}}\right)}{|y-t|^{n-r}} d\sigma_y\right] d\sigma_x - \int_\Omega \left[\int_{\Omega^x} \frac{\varphi(x,y)}{|y-t|^{n+2-r}|x-y|^{n-h}} d\sigma_y\right] d\sigma_x$$

$$+\lambda_6 \int_\Omega \left[\int_\Omega \frac{\Delta_x \varphi(x,y)}{|x-y|^{n-h}|y-t|^{n-r}} d\sigma_y\right] d\sigma_x + \lambda_6 \int_\Omega \left[\int_{\Omega^x} \frac{\Delta_x \varphi(x,y)}{|x-y|^{n-h}|y-t|^{n-r}} d\sigma_y\right] d\sigma_x$$

$$+\lambda_6 \int_\Omega \left[\int_\Omega \frac{(\partial_x \bar{\partial}_y + \partial_y \bar{\partial}_x)\varphi(x,y)}{|x-y|^{n-h}|y-t|^{n-r}} d\sigma_y\right] d\sigma_x$$

$$= \int_\Omega \left[\int_\Omega \frac{\varphi(x,y)}{|y-t|^{n+2-r}|x-y|^{n-h}} d\sigma_y\right] d\sigma_x$$

$$-\lambda_5 \int_\Omega \left[\int_{\Omega^t} \frac{\bar{\partial}_y \varphi(x,y)(\bar{x}-\bar{y}) + \partial_y \varphi(x,y)(x-y)}{|x-y|^{n+2-h}|y-t|^{n-r}} d\sigma_y\right] d\sigma_x$$

$$-\lambda_5 \int_\Omega \left[\int_{\Omega^t} \frac{\partial_y \left(\frac{\varphi(x,y)(x-y)}{|x-y|^{n+2-h}}\right) + \bar{\partial}_y \left(\frac{\varphi(x,y)(\bar{x}-\bar{y})}{|x-y|^{n+2-h}}\right)}{|y-t|^{n-r}} d\sigma_y\right] d\sigma_x$$

$$-\int_\Omega \left[\int_{\Omega^x} \frac{\varphi(x,y)}{|y-t|^{n+2-r}|x-y|^{n-h}} d\sigma_y\right] d\sigma_x + \lambda_6 \int_\Omega \left[\int_{\Omega^x} \frac{\Delta_x \varphi(x,y)}{|x-y|^{n-h}|y-t|^{n-r}} d\sigma_y\right] d\sigma_x$$

$$+\lambda_6 \int_\Omega \left[\int_\Omega \frac{\Delta_x \varphi(x,y) + (\partial_x \bar{\partial}_y + \partial_y \bar{\partial}_x)\varphi(x,y)}{|x-y|^{n-h}|y-t|^{n-r}} d\sigma_y\right] d\sigma_x.$$

证毕 .

第四章　第二类积分方程的可解性

积分方程问题是实际中非常重要的另一问题，Clifford 分析中积分方程的研究还是一个空白. 本章讨论了 Clifford 分析中第二类积分方程的可解性和解的表示式. M. Spivark 在他的专著 [18] 中，对 n 维实空间中流形上函数，借助于单位分解的方法给出了广义积分的定义，这个定义是一维实广义积分定义的推广，它允许在流形上出现多个奇点. 我们是在 Clifford 空间中用类似的方法定义了广义积分. Robert P. Gilbert 在文献 [2] 中引入了交换因子的概念，部分的解决了 Clifford 空间中乘积不可交换的问题. 借助于这一思想我们设计了带交换因子的积分算子核，并研究第二类带交换因子核的积分方程的可解性和解的级数表达式.

§4.1　问题的引入

在文献 [18] 中 M. Spivark 如下定义广义积分.

令 $D \subset R^{n+1}$ 是有界流形，θ 是 D 的开覆盖，若对每一个 $x \in D$，有 $U \in \theta$，使 $x \in U$，则 θ 称为 D 的可允许的开覆盖. 此时一定存在函数族 Φ，其中每一个函数 φ 在 D 上定义，属于 C^∞ 且满足以下条件

1. 对任一 $x \in D$，我们有 $0 \le \varphi(x) \le 1$.
2. 对任一 $x \in D$，存在一个开集 $V \subset R^{n+1}$ 包括点 x，并且只有有限个 $\varphi \in \Phi$ 在 V 上不等于 0.
3. 对每一个 $x \in D$，我们有 $\sum\limits_{\varphi \in \Phi} \varphi(x) = 1$.
4. 对每一个 $\varphi \in \Phi$，存在开集 U 属于 θ，使得 φ 在 U 内的某闭集外等于 0.

如果 $\Phi \subset C^\infty$ 满足 $1 \sim 3$，我们称 Φ 是 D 的一个单位分解. 如果 Φ 还满足 4，则称 Φ 从属于 θ 的单位分解.

设 Φ 是一个 D 上从属于 θ 的单位分解. f 是一个从 D 到 R 的函数，并且对 D 的任一点，存在其一个邻域使 f 在其上有界，集合 $\{x : f \text{ 在 } x \text{ 点不连续}\}$ 的测度为 0，并且每个 $\int_\Omega \varphi|f|$ 存在. 若 $\sum\limits_{\varphi \in \Phi} \int_\Omega \varphi|f|$ 收敛，则称 f 为 D 上的广义可积函数.

我们在 Clifford 分析中考虑函数族 $C_D(\mathcal{A})$, 其中函数定义在 D 上, 取值在 Clifford 空间.

定义 4.1 $f(x) \in C_D(\mathcal{A})$, 如果在 D 上存在一个可容许的开覆盖 θ 和一个从属于 θ 的单位分解 $\Phi(\subset C^{\infty})$, 使得每一个 $\int_D \varphi|f|$ 存在, 且 $\sum\limits_{\varphi\in\Phi} \int_D \varphi|f|$ 收敛 [10], 我们定义 f 为 M. Spivak 意义下的广义可积函数. $C_D(\mathcal{A})$ 中所有的广义可积函数可记作 $I_D(\mathcal{A})$.

下面的广义可积函数都是 M. Spivak 意义下的广义可积函数.

引理 4.1 $D \in R^n$ 如上, $f(x) = \Sigma f_A(x)e_A \in I_D(\mathcal{A}) \Longleftrightarrow$ 每一个 $f_A(x)$ 都是 D 上的广义可积函数.

引理 4.2 $D \in R^n$, $I_D(\mathcal{A})$ 如上, 则 $I_D(\mathcal{A})$ 对于加法和与 Cifford 数乘法封闭.

证明 令 $f, g \in I_D(\mathcal{A})$, 即在 D 上存在一个可容许的开覆盖 θ_1 和一个从属于 θ_1 的单位分解 Φ_1, 使得 f 在 D 上每一点的开集中有界, 并且每一个 $\int_D \varphi|f|$ 存在, $\sum\limits_{\varphi\in\Phi} \int_D \varphi|f|$ 收敛. 同时存在 D 上另一个可容许的开覆盖 θ_2 和一个从属于 θ_2 的单位分解 Φ_2, 使得 g 在 D 中每一点的开集上有界, 则每一 $\int_D \varphi|g|$ 存在且 $\sum\limits_{\varphi\in\Phi} \int_D \varphi|g|$ 收敛. 在 D 上做一个可容许的开覆盖 $\theta = \theta_1 \bigcup \theta_2 = \{U | U = u_1 \bigcup u_2, u_i \in \theta_i,\ i = 1, 2\}$ 和一个从属于 θ 的 D 上的单位分解 $\Phi = \{\varphi | \varphi = \frac{\varphi_1+\varphi_2}{2},\ \varphi_i \in \Phi_i,\ i = 1, 2\}$, 则

$$\sum_{\varphi\in\Phi} \int_D \varphi|f+g| \le \sum_{\varphi\in\Phi} \left[\int_D \varphi|f| + \int_D \varphi|g|\right]$$

$$\le \sum_{\varphi_1\in\Phi_1} \left[\int_D \varphi_1|f| + \int_D \varphi_1|g|\right] + \sum_{\varphi_2\in\Phi_2} \left[\int_D \varphi_2|f| + \int_D \varphi_2|g|\right],$$

右边的每一个积分都收敛, 所以左端积分也收敛. 即 $f+g$ 是广义可积的. 同理 αg (α 是 Clifford 数) 也是广义可积的. **证毕**.

定义 4.2 令 $f(x,y) = \sum\limits_A f_A(x,y)e_A$, $(x,y) \in D\times D$, 这里每一 $f_A(x,y) : D\times D \to R^1$ 是一个实函数. 如果 $f_A(x,y)$ 是每一变量的平方积分, 即 $|f|^2$ 看作 x 或 y 的函数 (其余变量看作常数) 是广义可积, 且 $\int_D \int_D |f(x,y)|^2 d_x d_y < \infty$, 则我们定义 $f(x,y)$ 为一个广义平方可积函数, 所有广义平方可积函数记作 $I_{D\times D}(\mathcal{A})$.

引理 4.3 令 $f(x,y)$, $g(x,y) \in I_{D\times D}(\mathcal{A})$ 且 $\Phi^2 \in I_D(\mathcal{A})$, 则

1. 关于 u (x, y 均看作常数)$f(x,u)g(u,y) \in I_D(\mathcal{A})$.
2. $\int_D f(x,u)g(u,y)du \in I_{D\times D}(\mathcal{A})$.
3. $\int_D f(x,u)\Phi^2(u)du \in I_D(\mathcal{A})$.

定义 4.3 令 $K(x,y) = \sum\limits_A K_A(x,y)e_A \in I_{D\times D}(\mathcal{A})$ 为一函数，定义相应的具有交换因子的核

$$K^0(x,y) = \sum_A K_A(x,y)e_A h_A,$$

这里 $h_A = h_{e_A} = h_{e_{r_1}\cdots e_{r_h}} = h_{e_{r_1}}\cdots h_{e_{r_h}}$, $A = \{r_1,\cdots,r_h\}$, 每一 h_{e_i} 是变换

$$h_{e_i}(e_j) = \begin{cases} e_j & i=j \\ -e_j & i\neq j \end{cases} \quad i,j = 1,\cdots,n,$$

我们称 h_A 是左交换因子.

定义 4.4 定义方程

$$\varphi(x) - \lambda\int_D K^0(x,u)\varphi(u)du = f(x) \tag{4.1}$$

为具有交换因子核的第二类型的积分方程. 这里 $\varphi(x)$ 是未知函数，$K^0(x,y)$ 如定义 4.3 所述，f 是已知函数，$f^2 \in I_D(\mathcal{A})$, $\lambda \in \mathcal{A}$ 是一个 Clifford 常数. 本章考虑寻找方程 (4.1) 的解.

§4.2 问题的解

使用逐次逼近的方法寻找方程 (4.1) 的解. 令

$$\begin{cases} \varphi_0(x) = f(x), \\ \varphi_m(x) = f(x) + \lambda\int_D K^0(x,u)\varphi_{m-1}(u)du, \ m = 1,2\cdots, \end{cases} \tag{4.2}$$

易知若能证明 $\{\varphi_m(x)\}$ D 上一致收敛于一个函数，则这个函数就是方程 (4.1) 的解.
为了考虑 $\{\varphi_m(x)\}$ 的性质，我们给出以下定义.

定义 4.5　$K(x,y)\in I_{D\times D}(\mathcal{A})$ 如上，我们定义

$$K_2(x,y)=\int_D K(x,t)K^0(t,y)dt,$$

$$K_m(x,y)=\int_D K(x,t)K_{m-1}(t,y)dt,\ m\geq 3,$$

作为 $K(x,y)$ 的 m 次叠加核， $m\geq 2$.

定理 4.1　如果 $K^0(x,y)$, $K(x,y)$, λ, $f(x)$ 如上所述，那么

$$\begin{cases} K^0(x,u)f(x)=f(x)K(x,u), \\ K^0(x,u)\lambda=\lambda K(x,u). \end{cases} \tag{4.3}$$

(根据交换因子的定义，此定理很容易证明).

分析函数序列 $\{\varphi_m(x)\}$

$$\begin{aligned}
\varphi_1(x) &= f(x)+\lambda\int_D K^0(x,t)\varphi_0(t)dt,\\
\varphi_2(x) &= f(x)+\lambda\int_D K^0(x,t)\varphi_1(t)dt\\
&= f(x)+\lambda\int_D K^0(x,t)\left[f(t)+\lambda\int_D K^0(t,u)f(u)du\right]dt\\
&= f(x)+\lambda\int_D K^0(x,u)f(u)\\
&\quad +(\lambda)^2\int_D\left[\int_D K(x,t)K^0(t,u)f(u)du\right]dt,\\
&= f(x)+\lambda\int_D K^0(x,u)f(u)+(\lambda)^2\int_D K_2(x,u)f(u)du,\\
\varphi_3(x) &= \cdots\cdots,
\end{aligned}$$

一般地，我们有

$$\begin{aligned}
\varphi_n(x) &= f(x)+\lambda\int_D K^0(x,u)f(u)du\\
&\quad +\sum_{m=2}^{n}(\lambda)^m\int_D K_m(x,u)f(u)du,\ n=1,2\cdots.
\end{aligned} \tag{4.4}$$

定理 4.2　令 $f(x)\in I_D(\mathcal{A})$, $\int_D|f(u)|^2dv_u=H^2$, $|K(x,u)|\leq M(u)$, 对每一 $x\in D, u\in D$ 均成立. $M(u)$ 是广义平方可积函数， $\int_D|M(u)|^2dv_u=L^2$, 这里 dv_u 指

D 上的体积元素，则

$$|\int_D K_m(x,u)f(u)du| \le HL^m J_1^m,\ m=1,2\cdots. \tag{4.5}$$

证明 当 $m=1$ 时，我们考虑 $K(x,u)=K^0(x,u)$ 且有

$$\left|\int_D K_1(x,u)f(u)du\right| \le \int_D |K_1(x,u)f(u)|dv_u \le J_1\int_D |M(u)|\,|f(u)|dv_u$$

$$\le J_1\left[\int_D |M(u)|^2dv_u\right]^{\frac{1}{2}}\left[\int_D |f(u)|^2dv_u\right]^{\frac{1}{2}} = J_1HL,$$

设 $m-1$ 时结论成立，我们证明 m 时结论也成立.

$$\left|\int_D K_m(x,u)f(u)du\right|$$

$$= \left|\int_D\int_D K(x,t)K_{(m-1)}(t,u)dtf(u)du\right|$$

$$= \left|\int_D K(x,t)[\int_D K_{(m-1)}(t,u)f(u)du]dt\right|$$

$$\le J_1\left[\int_D |K(x,t)|^2dv_t\right]^{\frac{1}{2}}\left[\int_D\int_D |K_{(m-1)}(t,u)f(u)du|^2dv_t\right]^{\frac{1}{2}}$$

$$\le J_1\left[\int_D |M(t)|^2dv_t\right]^{\frac{1}{2}}\left[\int_D J_1^{2(m-1)}H^2L^{2(m-1)}dv_t\right]^{\frac{1}{2}} \le HL^mJ_1^m.$$

证毕 .

定理 4.3 在定理 4.2 的条件下，对于满足 $|\lambda|<\frac{1}{J_1^2L}$ 的 Clifford 数 λ, 方程 (4.1) 有唯一解，且解为函数序列 (4.4) 的极限.

证明 由定理 4.2, 容易看出函数序列在 D 上一致收敛, 其极限函数就是方程 (4.1) 的解.

以下我们证明唯一性. 若存在 $\lambda \in \mathcal{A}$ 满足 $|\lambda| < \frac{1}{J_1^2 L}$, 有 $\varphi_1(x), \varphi_2(x)$ 是方程 (4.1) 的解，即

$$\varphi_1(x) - \lambda \int_D K^0(x,u)\varphi_1(u)du = f(x),$$

$$\varphi_2(x) - \lambda \int_D K^0(x,y)\varphi_2(u)du = f(x),$$

两式相减得 $\varphi_1(x) - \varphi_2(x) - \lambda \int_D K^0(x,u)[\varphi_1(u) - \varphi_2(u)]du = 0$, 令 $\omega(x) = \varphi_1(x) - \varphi_2(x)$, 我们有 $\omega(x) = \lambda \int_D K^0(x,u)\omega(u)du$, 因此

$$|\omega(x)|^2 \leq J_1^4 |\lambda|^2 \int_D |K^0(x,u)|^2 dv_u \int_D |\omega(u)|^2 dv_u,$$

在 D 上，不等式两边对 x 积分得

$$\int_D |\omega(x)|^2 dv_x J_1^4 \leq L^2 |\lambda|^2 \int_D |\omega(u)|^2 dv_u,$$

则有

$$(1 - J_1^4 L^2 |\lambda|^2) \int_D |\omega(u)|^2 dv_x \leq 0,$$

故

$$\int_D |\omega(u)|^2 dv_x = 0, \ \varphi_1 = \varphi_2.$$

证毕 .

定理 4.4　在定理 4.2 的条件下用等式 (4.4) 中的 $\{\varphi_n\}$ 替换方程 (4.1) 的准确解，则模的误差不超过

$$\frac{|\lambda|^{n+1} L^{n+1} H J_1^{2n+3}}{1 - |\lambda| L J_1^2}.$$

由以上推导知本定理易证.

第五章　双正则函数的非线性带位移边值问题

边值问题是在实际中应用较广的一类问题，本文以下几章讨论 Clifford 分析中的边值问题.

对照多复变函数理论 Bracks F. 和 Pincket W. 定义了 Clifford 分析中的双正则函数并讨论了双正则函数的 Cauchy 积分公式 [19],Le huang Son 讨论了 Clifford 分析中的 Hartogs 定理和 Cousin 问题 [5]. 徐振远、闻国椿、黄沙等研究了 Clifford 分析中的一系列边值问题 [20],[4],[6],[7],[9],[11],[12]. 在以上工作的基础上，第四章给出几个奇异积分算子，讨论其性质，进而研究了双正则函数的一个带位移的非线性边值问题解的存在性和积分表达式. 推广了文献 [4],[6],[7] 的工作.

§5.1　问　题　的　引　入

设 $D=D_1\times D_2$ 是 Euclidean 空间 $R^{m+1}\times R^{k+1}$, $1\le m\le n+1$, $1\le k\le n+1$ 中的一个连通开集， $D_1\subset R^{m+1}$, $D_2\subset R^{k+1}$. 我们研究 C^r 类函数集合

$$F_D^{(r)}=\left\{f\;\middle|\;\begin{array}{l} f:D\longrightarrow\mathcal{A},\ f(x,y)=\sum\limits_A f_A(x,y)e_A,\\ f_A(x,y)\in C^r(D),\ x\in R^{m+1},\ y\in R^{k+1}\end{array}\right\}.$$

对于 $f\in F_D^{(r)},(r\ge 1)$, 定义广义 Cauchy-Riemann 算子为 $\overline{\partial}_x f=\sum\limits_{i=0}^{m}\sum\limits_A e_ie_A\frac{\partial f_A}{\partial x_i}$, $f\overline{\partial}_y=\sum\limits_{j=0}^{k}\sum\limits_A e_Ae_j\frac{\partial f_A}{\partial y_j}$.

称方程组

$$\begin{cases}\overline{\partial}_x f=0\\ f\overline{\partial}_y=0\end{cases}$$

的解为双正则函数 [19].

设 D_i 的边界 Ω_i 均为光滑，定向紧致的 Liapunov 曲面， $(i=1,2)$[7], 讨论 $\Omega_1\times\Omega_2$

上的函数集合

$$H(\Omega_1\times\Omega_2,\beta)=\left\{\lambda(u,v)\;\middle|\;\begin{array}{l}\lambda(u,v)=\sum\limits_A\lambda_A e_A:\Omega_1\times\Omega_2\longrightarrow\mathcal{A},\\ |\lambda(u_1,v_1)-\lambda(u_2,v_2)|\le G|(u_1,v_1)-(u_2,v_2)|^\beta\end{array}\right\}.$$

其中 $0<\beta<1$, G 为正常数. $H(\Omega_1\times\Omega_2,\beta)$ 构成一 Banach 空间，范数为 $||\lambda||_\beta=C(\lambda,\Omega_1\times\Omega_2)+H(\lambda,\Omega_1\times\Omega_2,\beta)$. 其中第一项为连续模，第二项为 Hölder 模，并且有

$$||f+g||_\beta\le||f||_\beta+||g||_\beta,\quad ||fg||_\beta\le J_1||f||_\beta||g||_\beta, \tag{5.1}$$

其中 J_1 为正常数.

设 $D_i^+=D_i$, $i=1,2$, $D_1^-=R^{m+1}\backslash\overline{D}_1$, $D_2^-=R^{k+1}\backslash\overline{D}_2$. 当 x 从 $D_1^\pm$ 趋于 t_1 时记作 $x\longrightarrow t_1^\pm$, 当 y 从 $D_2^\pm$ 趋于 t_2 时记作 $y\longrightarrow t_2^\pm$, 当 $(x,y)\longrightarrow(t_1^{\pm\pm},t_2^{\pm\pm})$ 时，函数 $\Phi(x,y)$ 的极限记作 $\Phi^{\pm\pm}(x,y)$.

问题 R^* 求在 $D_1^+\times D_2^+$, $D_1^+\times D_2^-$, $D_1^-\times D_2^+$, $D_1^-\times D_2^-$ 内双正则连续到边界的函数 $\Phi(x,y)\in F_D^{(r)}$, 使其满足 $\Phi(\infty,\infty)=\Phi(x,\infty)=\Phi(\infty,y)=0$ 及带位移的边界条件

$$\begin{aligned}&A(t_1,t_2)\Phi^{++}(\alpha(t_1),t_2)+B(t_1,t_2)\Phi^{+-}(\alpha(t_1),t_2))+C(t_1,t_2)\Phi^{-+}(t_1,t_2)+\\&D(t_1,t_2)\Phi^{--}(t_1,t_2)=g(t_1,t_2)f(t_1,t_2,\Phi^{++},\Phi^{+-},\Phi^{-+},\Phi^{--}),\end{aligned} \tag{5.2}$$

其中 $\alpha(t_1):\Omega_1\longrightarrow\Omega_1$ 为一个 Haseman 位移 (见文献 [21]), 而 $A,B,C,D\in H(\Omega_1\times\Omega_2,\beta)$. 我们称以上问题为 **问题 R^***.

§5.2 Cauchy 型奇异积分及 Plemelj 公式

设 $\varphi\in H(\Omega_1\times\Omega_2,\beta)$, 考虑 Cauchy 型奇异积分

$$\alpha\int_{\Omega_1\times\Omega_2}E_1(t_1,u)d\sigma_u\varphi(u,v)d\sigma_v E_2(t_2,v), \tag{5.3}$$

其中 $E_1(l_1,l_2)=\dfrac{\bar{l}_2-\bar{l}_1}{|l_2-l_1|^{m+1}}$, $E_2(l_1,l_2)=\dfrac{\bar{l}_2-\bar{l}_1}{|l_2-l_1|^{k+1}}$ 称为积分核，而 $\alpha=\frac{1}{\omega_{m+1}\omega_{k+1}}$,

$\omega_{m+1},\omega_{k+1}$ 分别是空间 R^{m+1},R^{k+1} 中的单位超球的表面积.

定义 5.1 设 $(t_1,t_2)\in\Omega_1\times\Omega_2$, $\varphi\in H(\Omega_1\times\Omega_2,\beta)$. 作超球 $O_1:|x-t_1|<\delta$, $O_2:|y-t_2|<\delta$, 则 $O((t_1,t_2),\delta)=O_1\times O_2$ 为 (t_1,t_2) 的一个 δ 邻域, 记 $\Omega_1\times\Omega_2$ 位于 $O((t_1,t_2),\delta)$ 内部的部分为 λ_δ, 记

$$\Phi_\delta(t_1,t_2)=\alpha\int_{\Omega_1\times\Omega_2-\lambda_\delta}E_1(t_1,u)d\sigma_u\varphi(u,v)d\sigma_vE_2(t_2,v).$$

若 $\lim\limits_{\delta\to0^+}\Phi_\delta(t_1,t_2)$ 存在, 则称奇异积分 (5.3) 在主值意义下收敛, 其极限就记作 $\Phi(t_1,t_2)$. 其中 E_1, E_2 如上定义, 以下也类似.

为了研究上述奇异积分的性质, 我们再引入以下的 Cauchy 型积分.

定义 5.2 设 $\varphi\in H(\Omega_1\times\Omega_2,\beta)$, $x\in R^{(m+1)}\backslash\Omega_1$, $y\in R^{(k+1)}\backslash\Omega_2$, 称积分

$$\Phi(x,y)=\alpha\int_{\Omega_1\times\Omega_2}E_1(x,u)d\sigma_u\varphi(u,v)d\sigma_vE_2(y,v) \tag{5.4}$$

为 Cauchy 型积分.

引理 5.1 (由文献 [3] 知) 设 $\varphi\in H(\Omega_1\times\Omega_2,\beta)$, $x\in R^{(m+1)}\backslash\Omega_1$, $y\in R^{(k+1)}\backslash\Omega_2$, 则 (5.4) 式中的函数 $\Phi(x,y)$ 为双正则函数, 且 $\Phi(\infty,y)=\Phi(x,\infty)=\Phi(\infty,\infty)=0$.

引进空间 $H(\Omega_1\times\Omega_2,\beta)$ 积分算子 (参看文 [7])

$$P_1\varphi=\frac{2}{\omega_{m+1}}\int_{\Omega_1}E_1(t_1,u)d\sigma_u\varphi(u,t_2),$$

$$P_2\varphi=\frac{2}{\omega_{k+1}}\int_{\Omega_2}\varphi(t_1,v)d\sigma_vE_2(t_2,v),$$

$$P_3\varphi=4\Phi(t_1,t_2),\ \ P_i'=P_i(\alpha(t_1),t_2),\ \ i=1,2,3,$$

其中 $\alpha(t_1):\Omega_1\to\Omega_1$ 为 Haseman 位移.

引理 5.2 (见文献 [7]) 设 $\varphi(t_1,t_2)\in H(\Omega_1\times\Omega_2,\beta)$, 则对于 $(t_1,t_2)\in\Omega_1\times\Omega_2$, 有

$$\left.\begin{array}{lcl}\Phi^{++}(t_1,t_2)&=&\frac{1}{4}[\varphi+P_1\varphi+P_2\varphi+P_3\varphi](t_1,t_2)\\ \Phi^{+-}(t_1,t_2)&=&\frac{1}{4}[-\varphi-P_1\varphi+P_2\varphi+P_3\varphi](t_1,t_2)\\ \Phi^{-+}(t_1,t_2)&=&\frac{1}{4}[-\varphi+P_1\varphi-P_2\varphi+P_3\varphi](t_1,t_2)\\ \Phi^{--}(t_1,t_2)&=&\frac{1}{4}[\varphi-P_1\varphi-P_2\varphi+P_3\varphi](t_1,t_2)\end{array}\right\}. \tag{5.5}$$

推论 5.1 设 $\alpha(t_1),\varphi$ 和 $P_i, i=1,2,3$ 如上所设，则有

$$\left.\begin{aligned}\Phi^{++}(\alpha(t_1),t_2)&=\tfrac{1}{4}[\varphi+P_1\varphi+P_2\varphi+P_3\varphi](\alpha(t_1),t_2)\\ \Phi^{+-}(\alpha(t_1),t_2)&=\tfrac{1}{4}[-\varphi-P_1\varphi+P_2\varphi+P_3\varphi](\alpha(t_1),t_2)\end{aligned}\right\}. \tag{5.6}$$

即

$$\left.\begin{aligned}\Phi^{++}(\alpha(t_1),t_2)&=\tfrac{1}{4}[\varphi_1+P_1'\varphi+P_2'\varphi+P_3'\varphi](t_1,t_2)\\ \Phi^{+-}(\alpha(t_1),t_2)&=\tfrac{1}{4}[-\varphi_1-P_1'\varphi+P_2'\varphi+P_3'\varphi](t_1,t_2)\end{aligned}\right\}, \tag{5.7}$$

其中 $\varphi_1(t_1,t_2)=\varphi(\alpha(t_1),t_2)$.

§5.3 问题 R^* 的解的存在性

首先把问题转化成积分方程问题，假设问题 R^* 的解用 Cauchy 型积分 (5.4) 表示.

定理 5.1 问题 R^* 等价于求解积分方程

$$F\varphi=\varphi \quad \varphi(t_1,t_2)\in H(\Omega_1\times\Omega_2,\beta), \tag{5.8}$$

其中

$$\begin{aligned}F\varphi=&(A-B)(\varphi_1+P_1'\varphi)+(A+B)(P_2'\varphi+P_3'\varphi)\\&+(-C+D+1))\varphi+(C+D)(-P_2\varphi+P_3\varphi)+(C-D)P_1\varphi-4gf,\end{aligned} \tag{5.9}$$

其中 $\varphi_1, P_i, P_i', i=1,2,3$, 如上所述.

证明 将方程 (5.5) 和 (5.7) 代入边界条件 (5.2) 得

$$A(\varphi_1+P_1'\varphi+P_2'\varphi+P_3'\varphi)+B(-\varphi_1-P_1'\varphi+P_2'\varphi+P_3'\varphi)$$

$$+C(-\varphi+P_1\varphi-P_2\varphi+P_3\varphi)+D(\varphi-P_1\varphi-P_2\varphi+P_3\varphi)=4gf,$$

整理化简即为积分方程 (5.8). **证毕.**

引理 5.3 (见文献 [7]) 设 $\varphi(t_1,t_2)\in H(\Omega_1\times\Omega_2,\beta)$, 则存在与 φ 无关的常数 J_2 使得

$$\|\varphi\pm P_i\varphi\|_\beta\le J_2\|\varphi\|_\beta,\ \ \|P_i\varphi\|_\beta\le J_2\|\varphi\|_\beta,\ \ i=1,2,\ \ \|P_2\varphi\pm P_3\varphi\|_\beta\le J_2\|\varphi\|_\beta. \tag{5.10}$$

定理 5.2 设同胚映射 $\alpha(t_1):\Omega_1\to\Omega_1$ 满足 Lipschitz 条件

$$|(\alpha(t_{11}),t_{21})-(\alpha(t_{12}),t_{22})|\le H|(t_{11},t_{21})-(t_{12},t_{22})|, \tag{5.11}$$

并且 $\varphi\in H(\Omega_1\times\Omega_2,\beta)$, $\varphi_1(t_1,t_2)=\varphi(\alpha(t_1),t_2)$ ，则有

$$\|\varphi_1\|_\beta\le(1+H^\beta)\|\varphi\|_\beta,$$

这里 $H>1$ 为常数.

证明 设 $\varphi\in H(\Omega_1\times\Omega_2,\beta)$, 则对任意的 (t_{11},t_{21}), $(t_{12},t_{22})\in\Omega_1\times\Omega_2$ 有

$$|\varphi(t_{11},t_{21})|\le\|\varphi\|_\beta,$$

$$|\varphi(t_{11},t_{21})-\varphi(t_{12},t_{22})|\le\|\varphi\|_\beta|(t_{11},t_{21})-(t_{12},t_{22})|^\beta,$$

所以有

$$|\varphi_1(t_{11},t_{21})|=|\varphi(\alpha(t_{11}),t_{21})|=|\varphi((t'_{11}),t_{21})|\le\|\varphi\|_\beta,$$

其中 t'_{11} 为 Ω_1 上的某一点，并且有

$$\begin{aligned}&|\varphi_1(t_{11},t_{21})-\varphi_1(t_{12},t_{22})|=|\varphi(\alpha(t_{11}),t_{21})-\varphi((\alpha(t_{12}),t_{22})|\\ \le\;&\|\varphi\|_\beta|(\alpha(t_{11}),t_{21})-(\alpha(t_{12}),t_{22})|^\beta\le\|\varphi\|_\beta H^\beta|(t_{11},t_{21})-(t_{12},t_{22})|^\beta,\end{aligned}$$

综上有

$$\|\varphi_1\|_\beta\le(1+H^\beta)\|\varphi\|_\beta.$$

证毕 .

由引理 5.3 和定理 5.2 不难得到以下推论.

推论 5.2 设 $\varphi(t_1,t_2)\in H(\Omega_1\times\Omega_2,\beta)$, $\alpha(t_1)$, φ_1 如上所设，则存在与 φ 无关的常数 J_3 使得

$$\begin{aligned}&\|\varphi_1\pm P'_i\varphi\|_\beta\le J_3\|\varphi\|_\beta,\ \ \|P'_i\varphi\|_\beta\le J_3\|\varphi\|_\beta\ i=1,2,\\ &\|P'_2\varphi\pm P'_3\varphi\|_\beta\le J_3\|\varphi\|_\beta.\end{aligned} \tag{5.12}$$

将以上结果用于 Φ 的各边值表示式有以下推论.

推论 5.3 设推论 5.2 的条件满足，则有正常数 J_4 使得

$$\begin{aligned}&\|\Phi^{\pm\pm}(t_1,t_2)\|_\beta \le J_4\|\varphi\|_\beta,\ \ \|\Phi^{\mp\pm}(t_1,t_2)\|_\beta \le J_4\|\varphi\|_\beta,\\&\|\Phi^{\pm\pm}(\alpha(t_1),t_2)\|_\beta \le J_4\|\varphi\|_\beta,\ \ \|\Phi^{\mp\pm}(\alpha(t_1),t_2)\|_\beta \le J_4\|\varphi\|_\beta.\end{aligned} \tag{5.13}$$

定理 5.3 设 D_i 及其边界 Ω_i, $i=1,2$ 如上所述，$\alpha(t_1):\Omega_1\longrightarrow\Omega_1$ 为 Hasema 位移，从而是同胚映射，且满足 Lipschitz 条件 (5.11). 而 $f(t_1,t_2,\Phi^{(1)},\Phi^{(2)},\Phi^{(3)},\Phi^{(4)})$ 满足对 t 的 Hölder 条件和对 $\Phi^{(i)}, i=1,2,3,4$ 的 Lipschitz 条件，即

$$\begin{aligned}&|f(t_{11},t_{21},\Phi_1^{(1)},\Phi_1^{(2)},\Phi_1^{(3)},\Phi_1^{(4)})-f(t_{12},t_{22},\Phi_2^{(1)},\Phi_2^{(2)},\Phi_2^{(3)},\Phi_2^{(4)})|\\&\le J_5|(t_{11},t_{21})-(t_{12},t_{22})|^\beta+J_6|\Phi_1^{(1)}-\Phi_2^{(1)}|+\cdots+J_9|\Phi_1^{(4)}-\Phi_2^{(4)}|,\end{aligned} \tag{5.14}$$

其中 $J_i, i=5,6,7,8,9$ 为正常数. 又设 $f(0,0,0,0,0)=0$, A,B,C,D,g 都属于 $H(\Omega_1\times\Omega_2,\beta)$ 并且有 $\|A+B\|_\beta<h$, $\|B\pm D\|_\beta<h$, $\|D-C+1\|_\beta<h$, $0<h<1, \|g\|_\beta=\delta>0$, 若还有 $0<\delta<\frac{M(1-h(1+2J_2+2J_3))}{4J_1(J_{14}+J_{15}M)}$, 则问题 R^* 可解，其解的表达式如 (5.4) 式. 其中 M,J_{12},J_{13} 为常数.

证明 记连续函数空间 $C(\Omega_1\times\Omega_2)$ 的闭子集合 $H_M=\{\varphi|\varphi\in H(\Omega_1\times\Omega_2,\beta),\|\varphi\|_\beta\le M\}$.

首先证明算子 F 映 H_M 到自身. 由 F 的定义及定理的条件，对任意的 $\varphi\in H_M$ 有

$$\begin{aligned}&\|F\varphi\|_\beta\le J_1\|A-B\|_\beta\|\varphi_1+P_1'\|_\beta+J_1\|A+B\|_\beta\|P_2'\varphi+P_3'\varphi\|_\beta\\&+J_1\|-C+D+1\|_\beta\|\varphi\|_\beta+J_1\|C+D\|_\beta\|-P_2\varphi+P_3\varphi\|_\beta\\&+J_1\|C-D\|_\beta\|P_1\varphi\|_\beta+\|4gf\|_\beta\\&\le J_1h[\|\varphi_1+P_1'\varphi\|_\beta+\|P_2'\varphi+P_3'\varphi\|_\beta+\|\varphi\|_\beta+\|P_2\varphi-P_3\varphi\|_\beta+\|P_1\varphi\|_\beta]\\&+4J_1\|g\|_\beta\|f\|_\beta,\end{aligned}$$

由引理 5.3 和推论 5.2 有

$$\|F\varphi\|_\beta\le J_1h[1+2J_2+2J_3]\|\varphi\|_\beta+4J_1\|g\|_\beta\|f\|_\beta,$$

再由 (5.14) 式, 定理 5.2 和推论 5.2 有

$$\begin{aligned} C(f,\Omega_1\times\Omega_2) &= \max_{(t_1,t_2)\in\Omega_1\times\Omega_2}|f| \\ &= \max_{(t_1,t_2)\in\Omega_1\times\Omega_2}|f(t_1,t_2,\Phi^{++},\Phi^{+-},\Phi^{-+},\Phi^{--})-f(0,0,0,0,0,0)| \\ &\le J_5|(t_1,t_2)-(0,0)|^\beta+J_6|\Phi^{++}|+\cdots+J_9|\Phi^{--}| \\ &\le J_{10}+J_{11}\|\varphi\|_\beta, \end{aligned}$$

由此式和 (5.14) 式可得存在常数 J_{12},J_{13} 使得

$$H(f,\Omega_1\times\Omega_2,\beta)\le J_{12}+J_{13}\|\varphi\|_\beta,$$

所以存在常数 J_{14},J_{15} 使得

$$\|f\|_\beta\le J_{14}+J_{15}\|\varphi\|_\beta,$$

$$\|F\varphi\|_\beta\le J_1h[1+2J_2+2J_3]\|\varphi\|_\beta+4J_1\|g\|_\beta\|f\|_\beta]$$

$$\le J_1h[1+2J_2+2J_3]M+4J_1\delta[J_{11}+J_{12}M]<M,$$

所以 F 是映射 H_M 到自身的映射.

以下证 F 在 H_M 上连续, 对任意的函数列 $\{\varphi_n\}\in H_M$, 设 $\{\varphi_n\}$ 于 $\Omega_1\times\Omega_2$ 上一致收敛于 $\varphi\in H_M$, 即对任意的 $\varepsilon>0$ 存在 N, 当 $n>N$ 时有 $\|\varphi_n-\varphi\|_\beta<\varepsilon$. 由引理 5.3 和推论 5.2 可得

$$\begin{aligned} &\|\varphi_n-\varphi\pm P_i(\varphi_n-\varphi)\|_\beta\le J_2\|\varphi_n-\varphi\|_\beta<J_2\varepsilon,\ i=1,2,3, \\ &\|P_i(\varphi_n-\varphi)\|_\beta\le J_2\|\varphi_n-\varphi\|_\beta<J_2\varepsilon,\ i=1,2,3, \\ &\|P_2(\varphi_n-\varphi)\pm P_3(\varphi_n-\varphi)\|_\beta\le J_2\|\varphi_n-\varphi\|_\beta<J_2\varepsilon, \end{aligned}\tag{5.15}$$

以及

$$\begin{aligned} &\|(\varphi_{n1}-\varphi_1)\pm P_i(\varphi_{n1}-\varphi_1)\|_\beta\le J_3\|\varphi_n-\varphi\|_\beta<J_3\varepsilon,\ i=1,2,3, \\ &\|P_i'(\varphi_n-\varphi)\|_\beta\le J_3\|\varphi_n-\varphi\|_\beta<J_3\varepsilon,\ i=1,2,3, \\ &\|P_2'(\varphi_n-\varphi)\pm P_3'(\varphi_n-\varphi)\|_\beta,\le J_3\|\varphi_n-\varphi\|_\beta<J_3\varepsilon, \end{aligned}\tag{5.16}$$

其中 $\varphi_{n1}(t_1,t_2)=\varphi_n(\alpha(t_1),t_2)$. 由 (5.15), (5.16) 式有

$$\begin{aligned}
&|f(t_1,t_2,\Phi^{++}(\varphi_n),\Phi^{+-}(\varphi_n),\Phi^{-+}(\varphi_n),\Phi^{--}(\varphi_n))\\
&-f(t_1,t_2,\Phi^{++}(\varphi),\Phi^{+-}(\varphi),\Phi^{-+}(\varphi),\Phi^{--}(\varphi))|\\
&\le J_6|\Phi^{++}(\varphi_n)-\Phi^{++}(\varphi)|+J_7|\Phi^{+-}(\varphi_n)-\Phi^{+-}(\varphi)|+\\
&J_8|\Phi^{-+}(\varphi_n)-\Phi^{-+}(\varphi)|+J_9|\Phi^{--}(\varphi_n)-\Phi^{--}(\varphi)|\\
&=J_6|\Phi^{++}(\varphi_n-\varphi)|+J_7|\Phi^{+-}(\varphi_n-\varphi)|+J_8|\Phi^{-+}(\varphi_n-\varphi)|+J_9|\Phi^{--}(\varphi_n-\varphi)|\\
&\le (J_6J_4+J_7J_4+J_8J_4+J_9J_4)\|\varphi_n-\varphi\|_\beta=K\varepsilon,
\end{aligned}\tag{5.17}$$

由 (5.15), (5.16), (5.17) 知存在常数 G 使得

$$|F(\varphi_n-\varphi)|<G\varepsilon,$$

即 F 为 H_M 上的连续映射. 依据 Arzela-Ascoli 定理知, H_M 是连续空间 $C(\Omega_1\times\Omega_2)$ 中的紧集. 因此连续映射 F 映射 $C(\Omega_1\times\Omega_2)$ 中闭凸集 H_M 到自身, 并且 $F(H_M)$ 也是 $C(\Omega_1\times\Omega_2)$ 中的紧集. 再利用 Schauder 不动点原理知至少存在一个 $\varphi_0\in H(\Omega_1\times\Omega_2,\beta)$ 适合奇异积分方程 (5.8), 将 φ_0 代入 (5.4) 式得 $\Phi(x,y)$ 为问题 R^* 的解, 所以问题 R^* 至少存在一解, 并且以 (5.4) 为解的积分表达式. 其解 Φ 显然满足 $\Phi(\infty,\infty)=\Phi(x,\infty)=\Phi(\infty,y)=0$. **证毕** .

第六章　广义双正则函数的非线性带位移边值问题

§6.1 问 题 的 引 入

称方程组

$$\begin{cases} \overline{\partial}_x f = F_1 \\ f\overline{\partial}_y = F_2 \end{cases} \tag{6.1}$$

的解为广义双正则函数 [19].

设 $D_i^+ = D_i$, $i = 1, 2$, 分别为 R^{m+1}, 和 R^{k+1} 上的单位球, $D_1^- = R^{m+1}\backslash\overline{D}_1$, $D_2^- = R^{k+1}\backslash\overline{D}_2$, 当 x 从 $D_1^\pm$ 趋于 t_1 时记作 $x \longrightarrow t_1^\pm$, 当 y 从 $D_2^\pm$ 趋于 t_2 时记作 $y \longrightarrow t_2^\pm$, 当 $(x, y) \longrightarrow (t_1^{\pm\pm}, t_2^{\pm\pm})$ 时, 函数 $W(x, y)$ 的极限记作 $W^{\pm\pm}(t_1, t_2)$. 其中 $(t_1, t_2) \in \Omega_1 \times \Omega_2$, $\Omega_i = \partial D_i$, $i = 1, 2$ 为球的表面.

问题 $R^\triangle$　求在 $D_1^+ \times D_2^+$, $D_1^+ \times D_2^-$, $D_1^- \times D_2^+$, $D_1^- \times D_2^-$ 内广义双正则连续到边界的函数 $W(x, y) \in F_D^{(r)}$, 使其满足 $W(\infty, \infty) = W(x, \infty) = W(\infty, y) = 0$ 及带位移的边界条件

$$\begin{aligned} &A(t_1, t_2)W^{++}(\alpha(t_1), t_2) + B(t_1, t_2)W^{+-}(\alpha(t_1), t_2)) + C(t_1, t_2)W^{-+}(t_1, t_2)+ \\ &D(t_1, t_2)W^{--}(t_1, t_2) = g(t_1, t_2)f(t_1, t_2, W^{++}, W^{+-}, W^{-+}, W^{--}), \end{aligned} \tag{6.2}$$

其中 $\alpha(t_1) : \Omega_1 \longrightarrow \Omega_1$ 为一个 Haseman 位移 (见文献 [21]), 而 $A, B, C, D \in H(\Omega_1 \times \Omega_2, \beta)$ 及 f 都是已知函数. 我们称以上问题为 **问题** $R^\triangle$.

§6.2 广义双正则函数的 Plemelj 公式

给出形式算子

$$T_1F(x,y)=\frac{-1}{\omega_{m+1}}\int_{D_1^+}E_1(u,x)F(u,y)d\sigma_u-\frac{1}{\omega_{m+1}}\int_{D_1^+}E_3(u,x)F(\frac{1}{u},y)d\sigma_u,$$

$$T_2F(x,y)=\frac{-1}{\omega_{k+1}}\int_{D_2^+}F(x,v)E_2(v,y)d\sigma_v-\frac{1}{\omega_{k+1}}\int_{D_2^+}F(x,\frac{1}{v})E_4(v,y)d\sigma_v,$$

其中 E_1 和 E_2 如第五章第二节所述，而 $E_3(t_1,t_2)=\frac{\frac{1}{\bar{t}_1}-\bar{t}_2}{|\frac{1}{t_1}-t_2|^{m+1}|t_1|^{m+1}}$，$E_4(t_1,t_2)=\frac{\frac{1}{\bar{t}_1}-\bar{t}_2}{|\frac{1}{t_1}-t_2|^{k+1}|t_1|^{k+1}}$. 由文献 [2] 有以下结论.

引理 6.1 设对于每个固定的 $y\in R^{k+1}$，$F(x,y)\in L^{p,m+1}(R^{m+1})$，即 $|F(x,y)|$，$|F^{m+1}(x,y)|=|x|^{m+1}F(\frac{1}{x},y)\in L^p(D_1)$，并且其范数 $|F|_{p,m+1}=|F,D_1^+|_p+|F^{(m+1)},D_1^+|_p$ 与 y 无关，$p>m+1$，则对于每个固定的 $y\in R^{k+1}$ 有

1. $|T_1F|\le M(m+1,p)|F|_{p,m+1},\quad x\in R^{m+1}$.
2. 对于任意的 $x_1,x_2\in R^{m+1}$ 有

$$|T_1F(x_1,y)-T_1F(x_2,y)|\le M(m+1,p)|F|_{p,m+1}|x_1-x_2|^\alpha,\quad \alpha=\frac{p-m-1}{p}.$$

3. 对于 $|x|\ge 2$ 有 $|T_1F|\le M(m+1,p)|F|_{p,m+1}|x|^{(\frac{m+1}{p}-m)}$.
4. $\overline{\partial}_x(T_1F)=F,\quad x\in R^{m+1}$.

其中 $M(m+1,p)$ 为与 m,p 有关与 x,y 无关的正常数.

引理 6.2 设对于每个固定 $x\in R^{m+1}$，$F(x,y)\in L^{q,k+1}(R^{k+1})$. 即 $|F(x,y)|$，$|F^{k+1}(x,y)|=|y|^{k+1}F(x,\frac{1}{y})\in L^q(D_1)$，设其范数 $|F|_{q,k+1}=|F,D_2^+|_q+|F^{(k+1)},D_2^+|_q$ 与 x 无关，$q>k+1$，则对于每个固定 $x\in R^{m+1}$ 有

1. $|T_2F|\le M(k+1,q)|F|_{q,k+1},\quad y\in R^{k+1}$.

2. 对于任意 $y_1, y_2 \in R^{k+1}$ 有

$$|T_2F(x,y_1) - T_2F(x,y_2)| \le M(k+1,q)|F|_{q,k+1}|y_1 - y_2|^{\mu},\ \mu = \frac{q-k-1}{q}.$$

3. 对于 $|y| \ge 2$, 有 $|T_2F| \le M(k+1,q)|F|_{q,k+1}|y|^{(\frac{k+1}{q}-k)}$.

4. $(T_2F)\bar{\partial}_y = F, \quad y \in R^{k+1}$.

其中 $M(k+1,q)$ 为与 k,q 有关与 x,y 无关的正常数.

推论 6.1 在以上两个引理的条件下，另设 $q = \frac{p(k+1)}{m+1}$, 则

1. 对每个固定的 $y \in R^{m+1}$ 有 $T_1F \in C^{\alpha}(R^{m+1})$, 并且 $T_1F(\infty,y) = 0$, 其中 $\alpha = 1 - \frac{m+1}{p} = 1 - \frac{k+1}{q}$.
2. 对每个固定的 $x \in R^{m+1}$ 有 $T_2F \in C^{\alpha}(R^{k+1})$, 并且, $T_2F(x,\infty) = 0$, 其中 α 同上.

证明 由引理 6.1, 6.2, 注意到 $\mu = \frac{q-k-1}{q} = 1 - \frac{k+1}{q} = 1 - \frac{m+1}{p} = \alpha$, 并且注意到 $\frac{m+1}{p} - m < 1 - m \le 0$ 及 $\frac{k+1}{q} - k \le 0$ 即可. **证毕**.

定理 6.1 设 $F(x,y)$ 满足对于每个固定 $x \in R^{m+1}$, 任意 $y_1, y_2 \in R^{k+1}$, 有 $F(x,y) \in C^{\alpha}(R^{k+1}), 0 < \alpha < 1$, 并且, $|F(x,y_1) - F(x,y_2)| \le M_1|y_1 - y_2|^{\alpha}$ 其中 M_1 与 x 无关, $p > m+1, \alpha = 1 - \frac{m+1}{p}$, 则对于每个固定 $x \in R^{m+1}$, 有 $T_1F \in C^{\alpha}(R^{k+1})$.

证明 设 p' 适合 $\frac{1}{p} + \frac{1}{p'} = 1$, 因为 $p > m+1$, 故 $1 < p' < \frac{m+1}{m}$, $m < mp' < m+1$, 从而

$$\begin{aligned}
&|T_1F(x,y_1) - T_1F(x,y_2)| \\
= &\frac{1}{\omega_{m+1}}\left|\int_{D_1^+}\frac{\overline{u} - \overline{x}}{|u-x|^{m+1}}[F(u,y_1) - F(u,y_2)]d\sigma_u\right| \\
&+\frac{1}{\omega_{m+1}}\left|\int_{D_1^+}\frac{\frac{1}{\overline{u}} - \overline{x}}{|\frac{1}{u} - x|^{m+1}}[F(\frac{1}{u},y_1) - F(\frac{1}{u},y_2)]d\sigma_u\right|
\end{aligned}$$

$$\leq \frac{M_1}{\omega_{m+1}}\int_{D_1^+}|u-x|^{-m}|y_1-y_2|^{\alpha}|d\sigma_u|$$

$$+\frac{M_1}{\omega_{m+1}}\int_{D_1^+}|\frac{1}{u}-x|^{-m}|y_1-y_2|^{\alpha}|d\sigma_u|$$

$$\leq \frac{M_1}{\omega_{m+1}}\left[\int_{D_1^+}|u-x|^{-mp'}|d\sigma_u|\right]^{\frac{1}{p'}}\left||y_1-y_2|^{\alpha},D_1^+\right|_p$$

$$+\frac{M_1}{\omega_{m+1}}\left[\int_{D_1^+}|\frac{1}{u}-x|^{-mp'}|u|^{-(m+1)p'}|d\sigma_u|\right]^{\frac{1}{p'}}\left||y_1-y_2|^{\alpha},D_1^+\right|_p$$

$$\leq M_2|y_1-y_2|^{\alpha}.$$

证毕 .

定理 6.2 设 $F(x,y)$ 对于每个固定 $y\in R^{k+1}$，任意 x_1，$x_2\in R^{m+1}$，有 $F(x,y)\in C^{\alpha}(R^{m+1}),0<\alpha<1$，并且，$|F(x_1,y)-F(x_2,y)|\leq M_1|x_1-x_2|^{\alpha}$，其中 M_1 与 y 无关，$q>k+1,\alpha=1-\frac{k+1}{q}$，则对于每个固定 $y\in R^{k+1}$，有 $T_2F\in C^{\alpha}(R^{m+1})$.
(此定理的证明与定理 1 的证明类似，略.)

定理 6.3 设 $D_i,i=1,2$ 分别是 R^{m+1},R^{k+1} 中的单位超球，$F(x,y)$ 满足定理 6.1, 6.2 的条件，$p>m+1,q>k+1,\alpha=1-\frac{m+1}{p}=1-\frac{k+1}{q},q=\frac{p(k+1)}{m+1}$，则

$$T_iF(x,y)\in C^{\alpha}(R^{m+1},R^{k+1}),\ i=1,2.$$

证明 由题设，当 $x\in R^{m+1}$ 有 $F(x,y)\in L^{q,k+1}(R^{k+1})$，当 $y\in R^{k+1}$ 有 $F(x,y)\in$

$L^{p,m+1}(R^{m+1})$, 由以上的引理及定理，对任意的 $(x_1,y_1),(x_2,y_2)\in R^{m+1}\times R^{k+1}$ 有

$$|T_iF(x_1,y_1)-T_iF(x_2,y_2)|$$

$$\le|T_iF(x_1,y_1)-T_iF(x_2,y_1)+T_iF(x_2,y_1)-T_iF(x_2,y_2)|$$

$$\le M_3|x_1-x_2|^\alpha+M_4|y_1-y_2|^\alpha\le M_5|(x_1,y_1)-(x_2,y_2)|^\alpha,\ i=1,2.$$

证毕 .

定理 6.4 设 (6.1) 式中 $F_1,F_2\in C^\alpha(D)$,(其中 $D=D_1\times D_2$), 并且满足相容条件 $F_1\overline{\partial}_y=\overline{\partial}_xF_2$ 以及定理 6.3 中 F 所满足的条件, $F_1\overline{\partial}_y$ 也满足定理 3 中 F 所满足的条件, 则在 $D_1^\pm\times D_2^+$, $D_1^\pm\times D_2^-$ 中广义双正则函数 $W(x,y)$(方程 (6.1) 的解), 有如下积分表达式

$$W(x,y)=T_1F_1+T_2[F_2-(T_1F_1)\overline{\partial}_y]+\Phi(x,y),\tag{6.3}$$

其中 $\Phi(x,y)$ 是所属区域中的双正则函数.

证明 当 $(x,y)\in D_1^\pm\times D_2^+$, $D_1^\pm\times D_2^-$ 时, 由引理 6.1, 引理 6.2 知 $\overline{\partial}_x(T_1F_i)=F_i$, $(T_2F_i)\overline{\partial}_y=F_i$, $i=1,2$. 再由相容条件 $F_1\overline{\partial}_y=\overline{\partial}_xF_2$ 可得

$$\overline{\partial}_x\{T_1F_1+T_2[F_2-(T_1F_1)\overline{\partial}_y]\}=\overline{\partial}_xT_1F_1+\overline{\partial}_xT_2F_2-\overline{\partial}_xT_2[(T_1F_1)\overline{\partial}_y]$$

$$=F_1+T_2(\overline{\partial}_xF_2)-\overline{\partial}_xT_2[(T_1(F_1\overline{\partial}_y)]=F_1+T_2(\overline{\partial}_xF_2)-T_2[\overline{\partial}_x(T_1(F_1\overline{\partial}_y)]$$

$$=F_1+T_2(\overline{\partial}_xF_2)-T_2(F_1\overline{\partial}_y)=F_1+T_2(\overline{\partial}_xF_2)-T_2(\overline{\partial}_xF_2)=F_1,$$

并且

$$\{T_1F_1+T_2[F_2-(T_1F_1)\overline{\partial}_y]\}\overline{\partial}_y=(T_1F_1)\overline{\partial}_y+F_2-(T_1F_1)\overline{\partial}_y=F_2,$$

即 $T_1F_1 + T_2[F_2 - (T_1F_1)\overline{\partial}_y]$ 和 $W(x,y)$ 都是广义双正则函数, 所以

$$\{T_1F_1 + T_2[F_2 - (T_1F_1)\overline{\partial}_y]\} - W(x,y)$$

是双正则函数, 从而 (6.3) 式得证. **证毕**.

引进算子 $T_3(F_1\overline{\partial}_y) = T_2[(T_1F_1)\overline{\partial}_y]$.

定理 6.5 在定理 6.4 的条件下, 又设 $F_1(x,\infty) = F_2(\infty,y) = 0$, 则有

1. $\overline{\partial}_x[(T_3(F_1\overline{\partial}_y)] = T_2(F_1\overline{\partial}_y)$, $[T_3(F_1\overline{\partial}_y)]\overline{\partial}_y = T_1(F_1\overline{\partial}_y)$.
2. $T_3(F_1\overline{\partial}_y) \in C^\alpha(R^{m+1}, R^{k+1})$.
3. $T_3(F_1\overline{\partial}_y)(\infty,y) = T_3(F_1\overline{\partial}_y)(x,\infty) = T_3(F_1\overline{\partial}_y)(\infty,\infty) = 0$.

证明 从定理 6.4 的证明过程知道第一条成立, 利用定理 6.2 可知第二条成立, 由推论 6.1 知第三条成立. **证毕**.

定理 6.6 在定理 6.5 的条件下, 则有广义双正则函数的 Plemelj 公式

$$\begin{cases} W^{++} = T_1F_1 + T_2F_2 - T_3(F_1\overline{\partial}_y) + \frac{1}{4}[\varphi + P_1\varphi + P_2\varphi + P_3\varphi] \\ W^{+-} = T_1F_1 + T_2F_2 - T_3(F_1\overline{\partial}_y) + \frac{1}{4}[-\varphi - P_1\varphi + P_2\varphi + P_3\varphi] \\ W^{-+} = T_1F_1 + T_2F_2 - T_3(F_1\overline{\partial}_y) + \frac{1}{4}[-\varphi + P_1\varphi - P_2\varphi + P_3\varphi] \\ W^{--} = T_1F_1 + T_2F_2 - T_3(F_1\overline{\partial}_y) + \frac{1}{4}[\varphi - P_1\varphi - P_2\varphi + P_3\varphi] \end{cases} \tag{6.4}$$

在 $\Omega_1 \times \Omega_2$ 上对任意的 (t_1,t_2) 成立, 且 $W(\infty,y) = W(x,\infty) = W(\infty,\infty) = 0$.

证明 类似于第五章, 用 Cauchy 型积分表示 (6.3) 式中的 $\Phi(x,y)$, 再注意第五章的引理 5.2 、推论 5.1 知 (6.3) 式成立. 再由题设 $F_1(x,\infty) = F_2(\infty,y) = 0$ 可得 $T_1F_1(x,\infty) = T_2F_2(\infty,y) = 0$, 由定理 6.4, 6.5 知 $W(x,\infty) = W(\infty,y) = W(\infty,\infty) = 0$. **证毕**.

推论 6.2 若设 $\alpha(t_1):\Omega_1\longrightarrow\Omega_1$ 是一个 Haseman 位移，定理 6.5 的条件满足，则对任意的 $(t_1,t_2)\in\Omega_1\times\Omega_2$ 有

$$\left\{\begin{aligned} &W^{++}(\alpha(t_1),t_2)=\{T_1F_1+T_2F_2-T_3(F_1\overline{\partial}_y)\}(\alpha(t_1),t_2)\\ &\qquad\qquad+\tfrac{1}{4}[\varphi_1+P_1'\varphi+P_2'\varphi+P_3'\varphi],\\ \\ &W^{+-}(\alpha(t_1),t_2)=\{T_1F_1+T_2F_2-T_3(F_1\overline{\partial}_y)\}(\alpha(t_1),t_2)\\ &\qquad\qquad+\tfrac{1}{4}[-\varphi_1-P_1'\varphi+P_2'\varphi+P_3'\varphi],\end{aligned}\right. \tag{6.5}$$

其中 $\varphi_1(t_1,t_2)=\varphi(\alpha(t_1),t_2)$，而 $P_i'\varphi(t_1,t_2)=P_i\varphi(\alpha(t_1),t_2)$, $i=1,2,3$. P_j, $j=1,2,3$ 如第五章所述.

§6.3 问题 $R^{\triangle}$ 解的存在性

问题 $R^{\triangle}$ 的边界条件 (6.2) 可转化成奇异积分方程

$$L\varphi=\varphi, \tag{6.6}$$

其中 L 为空间 $H(\Omega_1\times\Omega_2,\beta)$ 上的奇异积分算子

$$\begin{aligned} &L\varphi=(A+B)(\varphi_1+P_1'\varphi+P_2'\varphi+P_3'\varphi)-2B(\varphi_1+P_1'\varphi)\\ &+(C+D)[-\varphi+P_1\varphi-P_2\varphi+P_3\varphi]+(2D+1))\varphi-2DP_1\varphi-4gf\\ &+4(A+B)[T_1F_1+T_2F_2-T_3(F_1\overline{\partial}_y)](\alpha(t_1),t_2)\\ &+4(C+D)[T_1F_1+T_2F_2-T_3(F_1\overline{\partial}_y)](t_1,t_2),\end{aligned} \tag{6.7}$$

其中 $\varphi_1,P_i,P_i',i=1,2,3$, 如第五章所述.

引理 6.3 设 D_i, Ω_i, $i=1,2$, 如上所述，在 $\Omega_1\times\Omega_2$ 上边界条件 (6.3) 式中的 f 满足

$$|f(t_{11},t_{21},W_1^{(1)},W_1^{(2)},W_1^{(3)},W_1^{(4)})-f(t_{12},t_{22},W_2^{(1)},W_2^{(2)},W_2^{(3)},W_2^{(4)})|$$

$$\le J_1|(t_{11},t_{21})-(t_{12},t_{22})|^\beta+J_2|W_1^{(1)}-W_2^{(1)}|+\cdots+J_5|W_1^{(4)}-W_2^{(4)}|),$$

其中 $J_i, i=1,2,3,4,5$ 是与 $W_j^m, j=1,2,\ m=1,2,3,4$ 无关的正常数，

$$|(t_{11},t_{21})-(t_{12},t_{22})|^\beta=(\sqrt{(t_{11}-t_{12})^2+(t_{21}-t_{22})^2}\ )^\beta.$$

又设 $f(0,0,0,0,0,0)=0$, 则存在常数 J_6, J_7 使

$$\|f\|_\beta\le J_6+J_7\|\varphi\|_\beta.$$

证明

$$\begin{aligned}
&C(f,\Omega_1\times\Omega_2)=\max_{(t_1,t_2)\in\Omega_1\times\Omega_2}|f|\\
=&\max_{(t_1,t_2)\in\Omega_1\times\Omega_2}|f(t_1,t_2,W^{++},W^{+-},W^{-+},W^{--}-f(0,0,0,0,0,0)|\\
\le&\ J_5|(t_1,t_2)-(0,0)|^\beta+J_6|\Phi^{++}|+\cdots+J_9|\Phi^{--}|\\
\le&\ J_8+J_9\|\varphi\|_\beta,
\end{aligned}$$

(此处 J_8, J_9 为正常数，由 (6.8) 式及 W 的有界性易知). 同时

$$|f(t_{11},t_{21},W^{++}(t_{11},t_{21}),W^{+-}(t_{11},t_{21}),W^{-+}(t_{11},t_{21}),W^{--}(t_{11},t_{21})$$

$$-f(t_{12},t_{22},W^{++}(t_{12},t_{22}),W^{+-}(t_{12},t_{22}),W^{-+}(t_{12},t_{22}),W^{--}(t_{12},t_{22}))|$$

$$\le(J_{10}+J_{11}\|\varphi\|_\beta)|(t_{11},t_{21})-(t_{12},t_{22})|^\beta.$$

综上所述有 $\|f\|_\beta\le J_6+J_7\|\varphi\|_\beta$. **证毕** .

引理 6.4 设 D_i, $i=1,2$, $\alpha(t)$ 如上所述， f 满足引理 6.3 的条件，另外假设

1. F_i, $i=1,2$, 满足定理 6.1, 6.2 关于 F 的条件, 相容条件 $F_1\overline{\partial}_y=\overline{\partial}_x F_2$ 及 $F_1(x,\infty)=F_2(\infty,y)=0$.
2. $T_1F_1, T_2F_2, T_3(F_1\overline{\partial}_y)$, $A,B,C,D\in H(\Omega_1\times\Omega_2,\beta), \alpha<\beta<1, \alpha=1-\frac{m+1}{p}$ 如推论 6.3 所述.
3. 另外设 $\gamma=J_{12}J_{13}[\|A+B\|_\beta+\|C+D\|_\beta+\|B\|_\beta+\|2D-1\|_\beta+2\|D\|_\beta]<1$（这里的 J_{12},J_{13} 都是正常数, 而 J_{12} 相当于第五章中的 J_1, J_{13} 相当于第五章中的 J_2, J_3, J_4 的最大值, 因为 A,B,C,D 为边界条件的系数, 所以这个条件可以满足）, 取 $\delta>0$ 使 $\delta<\frac{M(1-\gamma)}{4(\frac{1}{J_{13}}+J_{12}(J_6+J_7M))}$($M$ 为某正常数), $\|g\|_\beta<\delta$ 并 $\|T_1F_1+T_2F_2-T_3(F_1\overline{\partial}_y)\|_\beta<\delta$（$g,F_1,F_2$ 都为已知函数, 适当选取可使其符合要求）,

则有

1. 奇异算子 L 映 $C(\Omega_1\times\Omega_2)$ 的子空间 $T=\{\varphi|\varphi\in H(\Omega_1\times\Omega_2,\beta),\|\varphi\|_\beta<M\}$ 到自身.
2. 算子 L 是 T 上的连续映射.

证明

1. 由算子 L 的定义及关于模 $\|*\|_\beta$ 的性质有

$$
\begin{aligned}
&\|L\varphi\|_\beta\le J_{12}\|A+B\|_\beta\|\varphi_1+P_1'\varphi+P_2'\varphi+P_3'\varphi\|_\beta+J_{12}\|B\|_\beta\|2\varphi_1+2P_1'\varphi\|_\beta\\
&+J_{12}\|C+D\|_\beta\|-\varphi_1+P_1\varphi-P_2\varphi+P_3\varphi\|_\beta+J_{12}\|2D+1\|_\beta\|\varphi\|_\beta+J_{12}\|2D\|_\beta\|P_1\varphi\|_\beta\\
&+4J_{12}\|A+B+C+D\|_\beta\|T_1F_1+T_2F_2-T_3(F_1\overline{\partial}_y)\|_\beta+4J_{12}\|f\|_\beta\|g\|_\beta.
\end{aligned}
$$

由题设可得到

$$\|A+B+C+D\|_\beta\le\|A+B\|_\beta+\|C+D\|_\beta<\frac{1}{J_{12}J_{13}},$$

$$\|L\varphi\|_\beta\le J_{12}J_{13}\|\varphi\|_\beta[\|A+B\|_\beta+\|C+D\|_\beta+\|B\|_\beta+\|2D+1\|_\beta+2\|D\|_\beta]$$

$$+4J_{12}\tfrac{1}{J_{12}J_{13}}\delta+4J_{12}[J_6+J_7\|\varphi\|_\beta]\delta\le M\gamma+\delta[\tfrac{4}{J_{13}}+4J_{12}(J_6+J_7\|\varphi\|_\beta)]$$

$$\le M\gamma+M(1-\gamma)=M,$$

即算子 L 映 T 到自身.

2. 任取 $\varphi_n \in T$, $n = 1, 2, \cdots$, $\{\varphi_n\}$ 于 $\Omega_1 \times \Omega_2$ 上一致收敛于 φ, 任给 $\varepsilon > 0$, 当 n 充分大时 $||\varphi_n - \varphi||_\beta$ 可充分小, 由文 [3] 知当 n 充分大时, 对任意的 $(t_1, t_2) \in \Omega_1 \times \Omega_2$ 有, $|P_1\varphi_n - P_1\varphi| < \varepsilon$, $|P_2\varphi_n - P_2\varphi| < \varepsilon$, 类似的研究可得 $|P_3\varphi_n - P_3\varphi| < \varepsilon$, $|P_i'\varphi_n - P_i'\varphi| < \varepsilon$, $i = 1, 2, 3$, 总上可知 $|L\varphi_n - L\varphi| < \overline{W}\varepsilon$,($\overline{W}$ 为某正常数), 即 L 为 T 上的连续映射, **证毕** .

定理 6.7 在以上引理的条件下, 存在 $\varphi_0 \in T$ 使 $L\varphi_0 = \varphi_0$, 且问题 $R^\triangle$ 有解

$$W(x, y) = T_1F_1 + T_2[F_2 - (T_1F_1)\overline{\partial}_y] + \Phi(x, y), \tag{6.8}$$

其中

$$\Phi(x, y) = \frac{1}{\omega_{k+1}\omega_{m+1}} \int_{\Omega_1 \times \Omega_2} E_1(x, u) d\sigma_u \varphi_0(u, v) d\sigma_v E_2(v, y),$$

E_1, E_2 同文献同第五章.

证明 由引理 6.4 知 L 为 T 上连续到自身的映射, 由 Schauder 不动点原理知道存在 $\varphi_0 \in T$ 使 $L\varphi_0 = \varphi_0$, 将此 φ_0 代入上式得 $\Phi(x, y)$, 再将 $\Phi(x, y)$ 代入 (6.8) 式得到 $W(x, y)$, 则 $W(x, y)$ 为问题 $R^\triangle$ 的解, **证毕** .

第七章 Clifford 分析中无界域上双正则函数的 Plemelj 公式

近年来对 Clifford 分析中边值问题的研究多数局限于有界区域. 在有界域上对正则函数, 双正则函数, 广义正则函数研究了其 Plemelj 公式和一系列边值问题, 但是, 对于在无界域的情形, 却一直找不到较好的方法解决. 而在实际应用中, 许多问题都是在无界的情况下提出的, 因此讨论无界域上的 Plemelj 公式和边值问题有很重要的意义. E.Franks 和 J.Ryan[23] 发现了一种解决无界域上 Cauchy 型积分的方法, 即把原来的 Cauchy 核用一种新的核 K 代替, 虽然在代换后得到了许多有价值的结果, 但是, 这种代换只能讨论位于半空间内的无界区域. 1997 年, Klaus Gürlebeck,Uwe Kähler, John Ryan [24] 在文 [23] 的基础上, 对上述的核 K 进行了修正, 而且证明了引入修正的 Cauchy 核后, 原来在有界域上及引入核 K 后的一些重要结果仍然是成立的, 而且, 讨论不再局限于半空间内, 能够讨论任何补集中含有非空开集的无界域, 在这种修正核的意义下, 他们得到了一系列结果, 包括无界域上的 Cauchy 积分公式, 正则函数的 Plemelj 公式等. 本文就是在引入修正核的基础上, 给出了无界域上双正则函数 Cauchy 主值的存在性及具体表示式, 得到了双正则函数在无界域上的 Plemelj 公式, 推广了黄沙 [7] 和 Klaus Gürlebeck 等学者的工作.

§7.1 无界域上的 Cauchy 核和 Cauchy 积分公式

设 $D \subset R^{n+1}$ 是具有光滑 Liapunov 边界 Ω 的无界区域, 且 D 的余集包含一个非空开集, 我们选定一个位于 $\overline{D}$ 的余集中的点 u. 下面, 我们就来引入 D 的修正的 Cauchy 核.

定义 7.1 设 D 为上述无界区域, u 为 $\overline{D}$ 的余集中的点, $\xi \in \Omega$, $x \overline{\in} \Omega$, ω_{n+1} 为 R^{n+1} 中单位球面的面积, 称

$$l_n(\xi, x) = \frac{1}{\omega_{n+1}} \left(\frac{\bar{\xi} - \overline{x}}{|\xi - x|^{n+1}} - \frac{\bar{\xi} - \overline{u}}{|\xi - u|^{n+1}} \right) \tag{7.1}$$

为 n 维无界域 Ω 的 Cauchy 核.

可以证明，对这种修正的 Cauchy 核 $l_n(\xi,x)$ 有以下的定理.

引理 7.1 存在正常数 $J_1(n)>0$, 使得

$$|l_n(\xi,x)|\le J_1(n)|x-u|\sum_{j=1}^{n}|\xi-x|^{-j}|\xi-u|^{j-(n+1)}, \tag{7.2}$$

其中， $J_1(n)$ 是只与维数 n 有关的常数.

证明 由 Hile 引理 [2]

$$\begin{aligned}|l_n(\xi,x)| &\le \frac{J_1(n)\sum\limits_{k=0}^{n-1}|\xi-x|^{n-1-k}|\xi-u|^k}{|\xi-x|^n|\xi-u|^n}|x-u| \\ &= J_1(n)\sum_{k=0}^{n-1}|\xi-x|^{-1-k}|\xi-u|^{k-n}|x-u| \\ &= J_1(n)|x-u|\sum_{j=1}^{n}|\xi-x|^{-j}|\xi-u|^{j-(n+1)}.\end{aligned}$$

证毕 .

文 [24] 中给出了下面无界域上的 Cauchy 积分公式.

引理 7.2 (Cauchy **积分公式**)[24] D 如上所述， $f(x)$ 是 D 上的有界正则函数，且连续到边界，则对 $x\in D$ 有

$$f(x)=\int_{\Omega}l_n(\xi,x)\vec{n}(\xi)f(\xi)d\sigma(\xi), \tag{7.3}$$

这里， $\vec{n}(\xi)$ 是沿 Ω 的外法线方向的单位向量， $d\sigma(\xi)$ 是 Ω 上的 Lebesgue 测度.

§7.2 无界域上 Cauchy 型积分的 Cauchy 主值

设 $D=D_1\times D_2$ 是 Eucilidean 空间 $R^{m+1}\times R^{k+1}$ 中的无界连通开集， $D_i(i=1,2)$ 的余集中含有一个非空开集，设 $D_i(i=1,2)$ 的边界 $\Omega_i(i=1,2)$ 均为光滑，定向

的 Liapunov 曲面, $\Omega_1 \times \Omega_2$ 称为 $D_1 \times D_2$ 的特征边界, 同时引入 Hölder 连续函数空间 $H(\Omega_1 \times \Omega_2, s), 0 < s < 1$, 即若函数 $\varphi(\xi, \eta) \in H(\Omega_1 \times \Omega_2, s)$, 指函数 φ 分别对变量 $\xi \in \Omega_1, \eta \in \Omega_2$ 是 s 阶 Hölder 连续的. 下面就来讨论无界域上双正则函数及其 Cauchy 主值.

定理 7.1 设 $\Omega_i (i = 1, 2)$ 如上所述, u, v 分别属于 $R^{m+1}\backslash\overline{\Omega}_1$, $R^{k+1}\backslash\overline{\Omega}_2, x \overline{\in} \Omega_1, y \overline{\in} \Omega_2$, $\varphi(\xi, \eta)$ 是定义在 $\Omega_1 \times \Omega_2$ 上的有界的 Hölder 连续函数, 则

$$\Phi(x, y) = \int_{\Omega_1 \times \Omega_2} l_m(\xi, x) \vec{n}(\xi) d\sigma(\xi) \varphi(\xi, \eta) d\sigma(\eta) \vec{n}(\eta) l_k(\eta, y) \tag{7.4}$$

为双正则函数, 且 $\Phi(u, y) = \Phi(x, v) = \Phi(u, v) = 0$. 这里 ω_{m+1} 和 ω_{k+1} 分别为 R^{m+1} 和 R^{k+1} 中单位球面的表面积. $\vec{n}(\xi)$ 和 $\vec{n}(\eta)$ 分别为 Ω_1 和 Ω_2 上的外法线方向的单位向量. $l_m(\xi, x)$ 和 $l_k(\eta, y)$ 分别为定义 7.1 中所述的 m 维和 k 维无界空间的修正的 Cauchy 核.

证明 首先, 我们证明 $\Phi(x, y)$ 是有意义的. 任给 $(x, y) \overline{\in} \Omega_1 \times \Omega_2$, 由引理 7.1 知, 有

$$\begin{aligned}
|\Phi(x, y)| &\leq \int_{\Omega_1 \times \Omega_2} |l_m(\xi, x) \vec{n}(\xi) d\sigma(\xi) \varphi(\xi, \eta) d\sigma(\eta) \vec{n}(\eta) l_k(\eta, y)| \\
&\leq \int_{\Omega_1 \times \Omega_2} |l_m(\xi, x)| d\sigma(\xi) |\varphi(\xi, \eta)| d\sigma(\eta) |l_k(\eta, y)| \\
&\leq \int_{\Omega_1 \times \Omega_2} J_1(m) J_1(k) |x - u||y - v| \sum_{j=1}^{m} |\xi - x|^{-j} |\xi - u|^{j-(m+1)} \\
&\quad \cdot \sum_{j=1}^{k} |\eta - y|^{-j} |\eta - v|^{j-(k+1)} d\sigma(\xi) d\sigma(\eta).
\end{aligned}$$

由题设可知存在常数 M,N, 使对任意的 $\xi \in \Omega_1$, $\eta \in \Omega_2$ 有

$$|\xi - x| > M^{-1} |\xi - u|, \ |\eta - y| > N^{-1} |\eta - v|,$$

所以

$$|\Phi(x,y)| \le \int_{\Omega_1\times\Omega_2} J_1(m)J_1(k)|x-u||y-v|\sum_{j=1}^{m} M^j|\xi-u|^{-(m+1)}$$

$$\cdot\sum_{j=1}^{k} N^j|\eta-v|^{-(k+1)}d\sigma(\xi)d\sigma(\eta).$$

记 $\sum\limits_{j=1}^{m} M^j=\overline{M}$, $\sum\limits_{j=1}^{k} N^j=\overline{N}$, 则

$$|\Phi(x,y)| \le J_1(m)J_1(k)\overline{M}\,\overline{N}|x-u||y-v|\int_{\Omega_1\times\Omega_2}|\xi-u|^{-(m+1)}|\eta-v|^{-(k+1)}d\sigma(\xi)d\sigma(\eta).$$

由文 [24] 知上面的积分是存在的，因此， $\Phi(x,y)$ 存在. 以下，我们证明 $\Phi(x,y)$ 为双正则函数.

由文 [24] 知 $\overline{\partial}_x l_m(\xi,x)=l_k(\eta,y)\overline{\partial}_y=0$, 而积分号下的被积函数除 $l_m(\xi,x)$ 和 $l_k(\eta,y)$ 外均与 x,y 无关. 记被积函数为 $F(\xi,\eta,x,y)$, 显然有

$$\overline{\partial}_x F(\xi,\eta,x,y)=F(\xi,\eta,x,y)\overline{\partial}_y=0,$$

因此 $\int_{\Omega_1\times\Omega_2}\overline{\partial}_x F(\xi,\eta,x,y)d\sigma(\xi)d\sigma(\eta)=\int_{\Omega_1\times\Omega_2}F(\xi,\eta,x,y)\overline{\partial}_y d\sigma(\xi)d\sigma(\eta)=0$ 是关于 x,y 一致收敛的. 所以

$$\overline{\partial}_x\Phi(x,y)=\int_{\Omega_1\times\Omega_2}\overline{\partial}_x F(\xi,\eta,x,y)d\sigma(\xi)d\sigma(\eta)=0,$$

$$\Phi(x,y)\overline{\partial}_y=\int_{\Omega_1\times\Omega_2}F(\xi,\eta,x,y)\overline{\partial}_y d\sigma(\xi)d\sigma(\eta)=0,$$

由双正则函数的定义可知， $\Phi(x,y)$ 为双正则函数.

而由 $l_m(\xi,u)=l_k(\eta,v)=0$ 知 $\Phi(u,y)=\Phi(x,v)=\Phi(u,v)=0$. 证毕.

下面引入无界域上双正则函数的 Cauchy 主值的定义.

设 $(t_1,t_2)\in\Omega_1\times\Omega_2$, 记超球 $O_1:|x-t_1|<\delta,\ O_2:|y-t_2|<\delta$, 称 $O((t_1,t_2),\delta)=O_1\times O_2$ 为 (t_1,t_2) 的 δ 邻域，记 $\Omega_1\times\Omega_2$ 位于 $O((t_1,t_2),\delta)$ 内的部分为 λ_δ.

定义 7.2 称积分

$$\Phi(t_1,t_2)=\int_{\Omega_1\times\Omega_2}l_m(\xi,t_1)\vec{n}(\xi)d\sigma(\xi)\varphi(\xi,\eta)d\sigma(\eta)\vec{n}(\eta)l_k(\eta,t_2) \tag{7.5}$$

为无界特征流形 $\Omega_1\times\Omega_2$ 的奇异积分.

令 $\Phi(t_1,t_2)_\delta=\int_{\Omega_1\times\Omega_2\setminus\lambda_\delta}l_m(\xi,t_1)\vec{n}(\xi)d\sigma(\xi)\varphi(\xi,\eta)d\sigma(\eta)\vec{n}(\eta)l_k(\eta,t_2)$.

定义 7.3 若 $\lim\limits_{\delta\to 0}\Phi(t_1,t_2)_\delta=I(t_1,t_2)$ 存在,则称 $I(t_1,t_2)$ 为奇异积分 (7.5) 的 Cauchy 主值. 仍记作 $I=\Phi(t_1,t_2)$.

由文献 [24], 有以下定理.

引理 7.3 设 $a\in\mathcal{A}$ 为常数, 则对任意的 $t_i\in\Omega_i$, $i=1,2$ 有

$$\int_{\Omega_1}l_m(\xi,t_1)\vec{n}(\xi)d\sigma(\xi)a=\int_{\Omega_2}ad\sigma(\eta)\vec{n}(\eta)l_k(\eta,t_2)=\frac{1}{2}a. \tag{7.6}$$

为简便起见, 我们引入下列奇异积分算子

$$P_1\varphi=2\int_{\Omega_1}l_m(\xi,t_1)\vec{n}(\xi)d\sigma(\xi)\varphi(\xi,t_2),$$

$$P_2\varphi=2\int_{\Omega_2}\varphi(t_1,\eta)d\sigma(\eta)\vec{n}(\eta)l_k(\eta,t_2),$$

$$P_3\varphi=4\Phi(t_1,t_2).$$

定理 7.2 设 $\varphi(\xi,\eta)\in H(\Omega_1\times\Omega_2,s)$, 而且是 $\Omega_1\times\Omega_2$ 上的有界函数, 则奇异积分 (7.5) 的 Cauchy 主值是存在的, 而且

$$\Phi(t_1,t_2)=-\frac{1}{4}\varphi(t_1,t_2)+\chi(t_1,t_2)+\frac{1}{4}(P_1\varphi+P_2\varphi), \tag{7.7}$$

这里, $\chi(t_1,t_2)=\int_{\Omega_1\times\Omega_2}l_m(\xi,t_1)\vec{n}(\xi)d\sigma(\xi)\psi(\xi,\eta)d\sigma(\eta)\vec{n}(\eta)l_k(\eta,t_2)$,

$\psi(\xi,\eta)=\varphi(\xi,\eta)-\varphi(\xi,t_2)-\varphi(t_1,\eta)+\varphi(t_1,t_2)$.

证明 记 $\Phi(t_1,t_2)=I_1+I_2+I_3+I_4$,

$$I_1=\int_{\Omega_1\times\Omega_2} l_m(\xi,t_1)\vec{n}(\xi)d\sigma(\xi)\varphi(t_1,t_2)d\sigma(\eta)\vec{n}(\eta)l_k(\eta,t_2),$$

$$I_2=\int_{\Omega_1\times\Omega_2} l_m(\xi,t_1)\vec{n}(\xi)d\sigma(\xi)[\varphi(\xi,t_2)-\varphi(t_1,t_2)]d\sigma(\eta)\vec{n}(\eta)l_k(\eta,t_2),$$

$$I_3=\int_{\Omega_1\times\Omega_2} l_m(\xi,t_1)\vec{n}(\xi)d\sigma(\xi)[\varphi(t_1,\eta)-\varphi(t_1,t_2)]d\sigma(\eta)\vec{n}(\eta)l_k(\eta,t_2),$$

$$I_4=\int_{\Omega_1\times\Omega_2} l_m(\xi,t_1)\vec{n}(\xi)d\sigma(\xi)\psi(\xi,\eta)d\sigma(\eta)\vec{n}(\eta)l_k(\eta,t_2).$$

首先讨论 I_1, 由引理 7.3 知

$$\begin{aligned}&\int_{\Omega_1} l_m(\xi,t_1)\vec{n}(\xi)d\sigma(\xi)=\frac{1}{2},\\&\int_{\Omega_2} d\sigma(\eta)\vec{n}(\eta)l_k(\eta,t_2)=\frac{1}{2},\end{aligned}\tag{7.8}$$

所以, I_1 的 Cauchy 主值存在, 且 $I_1=\frac{1}{4}\varphi(t_1,t_2)$.

对 I_2, 由引理 7.3 和 (7.8) 式得

$$\begin{aligned}&\int_{\Omega_1\times\Omega_2} l_m(\xi,t_1)\vec{n}(\xi)d\sigma(\xi)[\varphi(\xi,t_2)-\varphi(t_1,t_2)]d\sigma(\eta)\vec{n}(\eta)l_k(\eta,t_2)\\=\;&\frac{1}{2}\int_{\Omega_1} l_m(\xi,t_1)\vec{n}(\xi)d\sigma(\xi)[\varphi(\xi,t_2)-\varphi(t_1,t_2)].\end{aligned}\tag{7.9}$$

适当选取以 $t_1\in\Omega_1$ 为心, 足够大的 r 为半径的超球 $U(t_1)$, 我们不妨选取这样的超球 $U(t_1)$, 使得当 $\xi\in\partial(D_1\setminus U(t_1))$ 时, $|\xi-t_1|>J_2^{-1}|\xi-u|, J_2$ 为常数. 由题设 t_1, u 为固定点知这是能够做到的, 则 (7.9) 可以写作

$$
\begin{aligned}
&\frac{1}{2}\int_{\partial(D_1\cap U_1)} l_m(\xi,t_1)\vec{n}(\xi)d\sigma(\xi)[\varphi(\xi,t_2)-\varphi(t_1,t_2)]\\
+\ &\frac{1}{2}\int_{\partial(D_1\setminus U_1)} l_m(\xi,t_1)\vec{n}(\xi)d\sigma(\xi)[\varphi(\xi,t_2)-\varphi(t_1,t_2)]\\
=\ &B_1+B_2.
\end{aligned}
$$

对 B_1, 由于其积分区域为一有界区域，把它写作

$$
\begin{aligned}
B_1\ =\ &\frac{1}{2}\int_{\partial(D_1\cap U_1)}\frac{1}{\omega_{m+1}}\left(\frac{\overline{\xi}-\overline{t}_1}{|\xi-t_1|^{m+1}}\vec{n}(\xi)d\sigma(\xi)[\varphi(\xi,t_2)-\varphi(t_1,t_2)]\right)\\
-\ &\frac{1}{2}\int_{\partial(D_1\cap U_1)}\frac{1}{\omega_{m+1}}\left(\frac{\overline{\xi}-\overline{u}}{|\xi-u|^{m+1}}\vec{n}(\xi)d\sigma(\xi)[\varphi(\xi,t_2)-\varphi(t_1,t_2)]\right).
\end{aligned}
$$

因为 $\varphi\in H(\Omega_1\times\Omega_2,s)$, 所以 $|\varphi(\xi,t_2)-\varphi(t_1,t_2)|\le J_7|\xi-t_1|^s$. 和文献 [7] 一样对第一个积分进行讨论，知它是存在的．而第二个积分为一正常积分，所以也是存在的.

对 B_2, 有

$$
\begin{aligned}
|B_2|\ =\ &|\frac{1}{2}\int_{\partial(D_1\setminus U_1)} l_m(\xi,t_1)\vec{n}(\xi)d\sigma(\xi)[\varphi(\xi,t_2)-\varphi(t_1,t_2)]|\\
\le\ &\frac{1}{2}\int_{\partial(D_1\setminus U_1)}|l_m(\xi,t_1)||\vec{n}(\xi)|d\sigma(\xi)|\varphi(\xi,t_2)-\varphi(t_1,t_2)|\\
\le\ &\frac{1}{2}\int_{\partial(D_1\setminus U_1)}2J_1(m)J_4\sum_{j=1}^{m}|\xi-t_1|^{-j}|\xi-u|^{j-(m+1)}|t_1-u|d\sigma(\xi)\\
\le\ &\frac{1}{2}\int_{\partial(D_1\setminus U_1)}2J_1(m)J_4J_5\frac{1}{|\xi-u|^{m+1}}|t_1-u|d\sigma(\xi)\\
=\ &J_1(m)J_4J_5\int_{\partial(D_1\setminus U_1)}\frac{1}{|\xi-u|^{m+1}}|t_1-u|d\sigma(\xi),
\end{aligned}
$$

这里， J_4 为 φ 在 $\Omega_1\times\Omega_2$ 上的最大值， $J_5=\sum_{j=1}^{m}J_2^j$.

由文献 [24] 知，这个积分是可积的，因而， B_2 是存在的.

由此， $I_2 \to \frac{1}{4}(P_1\varphi - \varphi)$.

同理，也可证明， I_3 是有意义的，为 $\frac{1}{4}(P_2\varphi - \varphi)$.

下证， I_4 的 Cauchy 主值也是存在的.

适当选取一以 t_1, t_2 为心的超球 $U(t_1), U(t_2)$, 其中 $U(t_1)$ 象前面一样选取， $U(t_2)$ 是使得当 $\eta \in \partial(D_2 \setminus U_2)$ 时，满足 $|\eta - t_2| > J_3^{-1}|\eta - v|$, J_3 为常数的超球，记

$$(\partial(D_1 \cap U_1)) \times (\partial(D_2 \cap U_2)) = \Omega_{11} \times \Omega_{21},$$

$$(\partial(D_1 \setminus U_1)) \times (\partial(D_2 \setminus U_2)) = \Omega_{12} \times \Omega_{22},$$

$$(\partial(D_1 \cap U_1)) \times (\partial(D_2 \setminus U_2)) = \Omega_{11} \times \Omega_{22},$$

$$(\partial(D_1 \setminus U_1)) \times (\partial(D_2 \cap U_2)) = \Omega_{12} \times \Omega_{21},$$

则区域 $\Omega_1 \times \Omega_2 = \Omega_{11} \times \Omega_{21} + \Omega_{12} \times \Omega_{22} + \Omega_{11} \times \Omega_{22} + \Omega_{12} \times \Omega_{21}$.

因此，$I_4(\Omega_1 \times \Omega_2) = I_4(\Omega_{11} \times \Omega_{21}) + I_4(\Omega_{12} \times \Omega_{22}) + I_4(\Omega_{12} \times \Omega_{21}) + I_4(\Omega_{11} \times \Omega_{22})$.

由于 $\Omega_{11} \times \Omega_{21}$ 为一有界区域，由文 [7] 知 $|\psi(\xi, \eta)| \leq J_7|\xi - t_1|^{\frac{\varepsilon}{2}}|\eta - t_2|^{\frac{\varepsilon}{2}}$. 和文 [7] 中的定理 2 一样的讨论知它的 Cauchy 主值是存在的. 即

$$I_4(\Omega_{11} \times \Omega_{21}) = \int_{\Omega_{11} \times \Omega_{21}} l_m(\xi, t_1)\vec{n}(\xi)d\sigma(\xi)\psi(\xi, \eta)d\sigma(\eta)\vec{n}(\eta)l_k(\eta, t_2).$$

对 $I_4(\Omega_{12} \times \Omega_{22})$ 有

$$
\begin{aligned}
&|I_4(\Omega_{12}\times\Omega_{22})| \\
&\le \int_{\Omega_{12}\times\Omega_{22}} |l_m(\xi,t_1)\vec{n}(\xi)d\sigma(\xi)\psi(\xi,\eta)d\sigma(\eta)\vec{n}(\eta)l_k(\eta,t_2)| \\
&\le \int_{\Omega_{12}\times\Omega_{22}} |l_m(\xi,t_1)|d\sigma(\xi)|\psi(\xi,\eta)|d\sigma(\eta)|l_k(\eta,t_2)| \\
&\le \int_{\Omega_{12}\times\Omega_{22}} J_1(m)J_1(k)|t_1-u|\sum_{j=1}^{m}|\xi-t_1|^{-j}|\xi-u|^{j-(m+1)} \\
&\quad \cdot \sup_{(\xi,\eta)\in(\Omega_1\times\Omega_2)}|\psi(\xi,\eta)||t_2-v|\sum_{j=1}^{k}|\eta-t_2|^{-j}|\eta-v|^{j-(k+1)}d\sigma(\xi)d\sigma(\eta) \\
&\le J_1(m)J_1(k)\sum_{j=1}^{m}J_2^j\sum_{j=1}^{k}J_3^j\int_{\Omega_{12}\times\Omega_{22}}|\xi-u|^{-(m+1)}|\eta-v|^{-(k+1)} \\
&\quad \cdot|t_1-u||t_2-v|d\sigma(\xi)d\sigma(\eta) \\
&\le J_1(m)J_1(k)J_5J_6\int_{\Omega_{12}\times\Omega_{22}}|\xi-u|^{-(m+1)}|\eta-v|^{-(k+1)} \\
&\quad \cdot|t_1-u||t_2-v|d\sigma(\xi)d\sigma(\eta),
\end{aligned}
$$

这里，$J_5=\sum_{j=1}^{m}J_2^j, J_6=\sum_{j=1}^{k}J_3^j$ 为正常数. 上面的积分是存在的，所以有

$$
I_4(\Omega_{12}\times\Omega_{22})|=\int_{\Omega_{12}\times\Omega_{22}} l_m(\xi,t_1)\vec{n}(\xi)d\sigma(\xi)\psi(\xi,\eta)d\sigma(\eta)\vec{n}(\eta)l_k(\eta,t_2).
$$

由区域的构造可知，对 $I_4(\Omega_{12}\times\Omega_{21})$ 和 $I_4(\Omega_{11}\times\Omega_{22})$, 我们只讨论其中的一个即可. 因为 $|\psi(\xi,\eta)|\le J_7|\xi-t_1|^s$ 所以

$$
\begin{aligned}
|I_4(\Omega_{11}\times\Omega_{22})| &\le \int_{\Omega_{12}\times\Omega_{22}} |l_m(\xi,t_1)\vec{n}(\xi)d\sigma(\xi)\psi(\xi,\eta)d\sigma(\eta)\vec{n}(\eta)l_k(\eta,t_2)| \\
&\le \int_{\Omega_{11}\times\Omega_{22}} |l_m(\xi,t_1)|d\sigma(\xi)|\psi(\xi,\eta)|d\sigma(\eta)|l_k(\eta,t_2)|
\end{aligned}
$$

$$\leq \int_{\Omega_{11}\times\Omega_{22}} \frac{|\psi(\xi,\eta)|}{|\xi-t_1|^m}|l_k(\eta,t_2)|d\sigma(\xi)d\sigma(\eta)$$

$$+\int_{\Omega_{11}\times\Omega_{22}} \frac{|\psi(\xi,\eta)|}{|\xi-u|^m}|l_k(\eta,t_2)|d\sigma(\xi)d\sigma(\eta)$$

$$\leq \int_{\Omega_{11}} \frac{J_7}{|\xi-t_1|^{m-s}}d\sigma(\xi)\int_{\Omega_{22}} J_1(k)|t_2-v|J_6|\eta-u|^{-(k+1)}d\sigma(\eta)$$

$$+\int_{\Omega_{11}} \frac{J_4}{|\xi-t_1|^m}d\sigma(\xi)\int_{\Omega_{22}} J_1(k)|t_2-v|J_6|\eta-u|^{-(k+1)}d\sigma(\eta),$$

由文 [24] 和文 [7] 知上面的积分是存在的，所以

$$I_4(\Omega_{11}\times\Omega_{22}) = \int_{\Omega_{11}\times\Omega_{22}} l_m(\xi,t_1)\vec{n}(\xi)d\sigma(\xi)\psi(\xi,\eta)d\sigma(\eta)\vec{n}(\eta)l_k(\eta,t_2).$$

综上所述，I_4 的 Cauchy 主值是存在的．而且其积分主值为 $\chi(t_1,t_2)$.

综上，奇异积分 (7.5) 的 Cauchy 主值是存在的，为

$$-\frac{1}{4}\varphi(t_1,t_2)+\chi(t_1,t_2)+\frac{1}{4}(P_1\varphi+P_2\varphi).$$

证毕．

§7.3 无界区域上双正则函数的 Plemelj 公式

定理 7.3 Ω_1,Ω_2 为如上述的无界区域 D_1,D_2 的边界，$l_m(\xi,x),l_m(\xi,t_1),l_k(\eta,y)$, $l_k(\eta,t_2)$ 如上，$\varphi(\xi,\eta)\in H(\Omega_1\times\Omega_2,s),(x,y)\overline{\in}\Omega_1\times\Omega_2$, 且 $\varphi(\xi,\eta)$ 是 $\Omega_1\times\Omega_2$ 上的有界函数，则

$$\lim_{(x,y)\to(t_1,t_2)}\chi(x,y)=\chi(t_1,t_2),\quad (t_1,t_2)\in\Omega_1\times\Omega_2,$$

这里，

$$\chi(x,y)=\int_{\Omega_1\times\Omega_2} l_m(\xi,x)\vec{n}(\xi)d\sigma(\xi)\psi(\xi,\eta)d\sigma(\eta)\vec{n}(\eta)l_k(\eta,y), \tag{7.10}$$

$\psi(\xi,\eta)=\varphi(\xi,\eta)-\varphi(\xi,t_2)-\varphi(t_1,\eta)+\varphi(t_1,t_2).$

证明

$$
\begin{aligned}
&\chi(x,y)-\chi(t_1,t_2)\\
=\ &\int_{\Omega_1\times\Omega_2} l_m(\xi,x)\vec{n}(\xi)d\sigma(\xi)\psi(\xi,\eta)d\sigma(\eta)\vec{n}(\eta)l_k(\eta,y)\\
&-\int_{\Omega_1\times\Omega_2} l_m(\xi,t_1)\vec{n}(\xi)d\sigma(\xi)\psi(\xi,\eta)d\sigma(\eta)\vec{n}(\eta)l_k(\eta,t_2)\\
=\ &\int_{\Omega_1\times\Omega_2}[l_m(\xi,x)-l_m(\xi,t_1)]\vec{n}(\xi)d\sigma(\xi)\psi(\xi,\eta)d\sigma(\eta)\vec{n}(\eta)l_k(\eta,t_2)\\
&+\int_{\Omega_1\times\Omega_2}[l_m(\xi,x)-l_m(\xi,t_1)]\vec{n}(\xi)d\sigma(\xi)\psi(\xi,\eta)d\sigma(\eta)\vec{n}(\eta)[l_k(\eta,y)-l_k(\eta,t_2)]\\
&+\int_{\Omega_1\times\Omega_2} l_m(\xi,t_1)\vec{n}(\xi)d\sigma(\xi)\psi(\xi,\eta)d\sigma(\eta)\vec{n}(\eta)[l_k(\eta,y)-l_k(\eta,t_2)]\\
=\ &L_1+L_2+L_3.
\end{aligned}
$$

为简便起见，我们只讨论 L_1 即可，至于 L_2, L_3，其讨论的过程和 L_1 类似.

$L_1=\int_{\Omega_1\times\Omega_2}[l_m(\xi,x)-l_m(\xi,t_1)]\vec{n}(\xi)d\sigma(\xi)\psi(\xi,\eta)d\sigma(\eta)\vec{n}(\eta)l_k(\eta,t_2).$

象定理 7.2 一样选取超球 $U(t_1), U(t_2)$，并象定理 7.2 一样将区域 $\Omega_1\times\Omega_2$ 分成 4 部分，则显然有

$$
\begin{aligned}
L_1(\Omega_1\times\Omega_2) &= L_1(\Omega_{11}\times\Omega_{21})+L_1(\Omega_{12}\times\Omega_{22})\\
&\quad +L_1(\Omega_{11}\times\Omega_{22})+L_1(\Omega_{11}\times\Omega_{21}).
\end{aligned}
\tag{7.11}
$$

对 $L_1(\Omega_{11}\times\Omega_{21})$，注意到 $l_m(\xi,x)-l_m(\xi,t_1)=\frac{1}{\omega_{m+1}}\left(\frac{\overline{\xi}-\overline{x}}{|\xi-x|^{m+1}}-\frac{\overline{\xi}-\overline{t}_1}{|\xi-t_1|^{m+1}}\right)$，由定理 7.2 和文献 [23] 的定理 3 知，当 $(x,y)\to(t_1,t_2)$ 时，$L_1(\Omega_{11}\times\Omega_{21})\to 0$.

对于 $L_1(\Omega_{12}\times\Omega_{22})$，有

$$|L_1(\Omega_{12}\times\Omega_{22})| = |\int_{\Omega_{12}\times\Omega_{22}}[l_m(\xi,x)-l_m(\xi,t_1)]\vec{n}(\xi)d\sigma(\xi)\psi(\xi,\eta)d\sigma(\eta)\vec{n}(\eta)l_k(\eta,t_2)|$$

$$\leq \int_{\Omega_{12}\times\Omega_{22}} J_1(m)J_1(k)\sum_{j=1}^{m}|\xi-x|^{-j}|\xi-t_1|^{j-(m+1)}|x-t_1||\psi(\xi,\eta)|$$
$$\cdot\sum_{j=1}^{k}|\eta-t_2|^{-j}|\eta-v|^{j-(k+1)}|t_2-v|d\sigma(\xi)d\sigma(\eta).$$

和定理 7.2 一样的讨论得到

$$|L_1(\Omega_{12}\times\Omega_{22})| \leq |x-t_1|J_8\int_{\Omega_{12}\times\Omega_{22}}\frac{d\sigma(\xi)}{|\xi-t_1|^{m+1}}\frac{d\sigma(\eta)}{|\eta-v|^{k+1}}, \qquad (7.12)$$

这里 $J_8 = J_4J_5J_6J_1(m)J_1(k)$ 为正常数.

这个式子右边的积分是可积的，因此，对任何 $(t_1,t_2)\in\Omega_1\times\Omega_2$, 当 $(x,y)\to(t_1,t_2)$ 时， $L_1(\Omega_{12}\times\Omega_{22})\to 0$.

下面讨论 $L_1(\Omega_{11}\times\Omega_{22})$, $L_1(\Omega_{12}\times\Omega_{21})$, 我们只讨论其中的一个即可，另一个类似可得.

$$\begin{aligned}
&L_1(\Omega_{12}\times\Omega_{21})\\
=\ &\int_{\Omega_{12}\times\Omega_{21}}[l_m(\xi,x)-l_m(\xi,t_1)]\vec{n}(\xi)d\sigma(\xi)\psi(\xi,\eta)d\sigma(\eta)\vec{n}(\eta)l_k(\eta,t_2)\\
=\ &\int_{\Omega_{12}\times\Omega_{21}}[l_m(\xi,x)-l_m(\xi,t_1)]\vec{n}(\xi)d\sigma(\xi)\psi(\xi,\eta)d\sigma(\eta)\vec{n}(\eta)\frac{\overline{\eta}-\overline{t}_2}{|\eta-t_2|^{k+1}}\\
&-\int_{\Omega_{12}\times\Omega_{21}}[l_m(\xi,x)-l_m(\xi,t_1)]\vec{n}(\xi)d\sigma(\xi)\psi(\xi,\eta)d\sigma(\eta)\vec{n}(\eta)\frac{\overline{\eta}-\overline{v}}{|\eta-v|^{k+1}}\\
=\ &B_1-B_2,
\end{aligned}$$

所以， $|L_1(\Omega_{12}\times\Omega_{21})|\leq|B_1|+|B_2|$.

由文 [7] 知， $|\psi(\xi,\eta)|\leq J_7|\eta-t_2|^s$, 所以，有

$$|B_1| \le \int_{\Omega_{12}\times\Omega_{21}} |l_m(\xi,x) - l_m(\xi,t_1)\vec{n}(\xi)d\sigma(\xi)\psi(\xi,\eta)d\sigma(\eta)\vec{n}(\eta)\frac{\overline{\eta}-\overline{t}_2}{|\eta-t_2|^{k+1}}|$$

$$\le \int_{\Omega_{12}} |x-t_1|J_5J_1(m)\frac{d\sigma(\xi)}{|\xi-t_1|^{m+1}}\int_{\Omega_{21}} J_7\frac{d\sigma(\eta)}{|\eta-t_2|^{k-s}},$$

所以，当 $(x,y)\to(t_1,t_2)$ 时， $|B_1|\to 0$.

而 $|B_2| \le \int_{\Omega_{12}} |x-t_1|J_5J_1(m)\frac{d\sigma(\xi)}{|\xi-t_1|^{m+1}}\sup_{(\xi,\eta)\in\Omega_1\times\Omega_2}|\psi(\xi,\eta)|\int_{\Omega_{21}}\frac{d\sigma(\eta)}{|\eta-v|^k}$，所以，当 $(x,y)\to(t_1,t_2)$ 时， $|B_2|\to 0$. 因此， $L_1(\Omega_{12}\times\Omega_{21})\to 0$.

同理可知，当 $(x,y)\to(t_1,t_2)$ 时， $L_1(\Omega_{11}\times\Omega_{22})\to 0$. 于是，$\lim\limits_{(x,y)\to(t_1,t_2))}\chi(x,y) = \chi(t_1,t_2)$. 证毕 .

记 $D_i^+ = D_i$, $(i=1,2)$, $D_1^- = R^{m+1}\setminus\overline{D}_1$, $D_2^- = R^{k+1}\setminus\overline{D}_2$. 设 (x,y) 从 $D_1^\pm\times D_2^\pm$ 趋于 $(t_1,t_2)\in\Omega_1\times\Omega_2$ ，记作 $(x,y)\to(t_1^\pm,t_2^\pm)$. 当 $(x,y)\to(t_1^\pm,t_2^\pm)$ 时，Cauchy 型积分 (7.4) 的极限值对应的记作 $\Phi^{\pm\pm}(t_1,t_2)$ ，我们就得到了无界域上双正则函数的 Plemelj 公式.

定理 7.4 设 D_1,D_2 均为上述的区域，算子 P_1,P_2,P_3 分别为前面所述，(7.4) 式中的 $\varphi(\xi,\eta)\in H(\Omega_1\times\Omega_2,s)$, 且 $\varphi(\xi,\eta)$ 是 $\Omega_1\times\Omega_2$ 上的有界函数，则对 $(t_1,t_2)\in\Omega_1\times\Omega_2$, 有

$$\left.\begin{aligned}
\Phi^{++}(t_1,t_2) &= \frac{1}{4}[\varphi(t_1,t_2)+P_1\varphi+P_2\varphi+P_3\varphi]\\
\Phi^{+-}(t_1,t_2) &= \frac{1}{4}[-\varphi(t_1,t_2)-P_1\varphi+P_2\varphi+P_3\varphi]\\
\Phi^{-+}(t_1,t_2) &= \frac{1}{4}[-\varphi(t_1,t_2)+P_1\varphi-P_2\varphi+P_3\varphi]\\
\Phi^{--}(t_1,t_2) &= \frac{1}{4}[\varphi(t_1,t_2)-P_1\varphi-P_2\varphi+P_3\varphi]
\end{aligned}\right\}. \tag{7.13}$$

证明 改写 Cauchy 型积分 (7.4) 为

$$\begin{aligned}\Phi(x,y) &= \chi(x,y)+\int_{\Omega_1\times\Omega_2} l_m(\xi,x)\vec{n}(\xi)d\sigma(\xi)\varphi(t_1,t_2)d\sigma(\eta)\vec{n}(\eta)l_k(\eta,y)\\ &+\int_{\Omega_1\times\Omega_2} l_m(\xi,x)\vec{n}(\xi)d\sigma(\xi)[\varphi(\xi,t_2)-\varphi(t_1,t_2)]d\sigma(\eta)\vec{n}(\eta)l_k(\eta,y)\\ &+\int_{\Omega_1\times\Omega_2} l_m(\xi,x)\vec{n}(\xi)d\sigma(\xi)[\varphi(t_1,\eta)-\varphi(t_1,t_2)]d\sigma(\eta)\vec{n}(\eta)l_k(\eta,y).\end{aligned}$$

由定理 7.2, 定理 7.3 知

$$\chi^{\pm\pm}(t_1,t_2)=\chi^{\mp\pm}(t_1,t_2)=\chi(t_1,t_2)=\frac{1}{4}[\varphi-P_1\varphi-P_2\varphi+P_3\varphi].$$

再由文献 [24] 中的定理 3.3, 定理 3.4 以及前面 Cauchy 积分公式可得结论成立.

证毕.

推论 7.1 条件如上, 则

$$\left.\begin{aligned}\Phi^{++}(t_1,t_2)-\Phi^{+-}(t_1,t_2)-\Phi^{-+}(t_1,t_2)+\Phi^{--}(t_1,t_2)&=\varphi(t_1,t_2)\\ \Phi^{++}(t_1,t_2)-\Phi^{+-}(t_1,t_2)+\Phi^{-+}(t_1,t_2)-\Phi^{--}(t_1,t_2)&=P_1\varphi\\ \Phi^{++}(t_1,t_2)+\Phi^{+-}(t_1,t_2)-\Phi^{-+}(t_1,t_2)-\Phi^{--}(t_1,t_2)&=P_2\varphi\\ \Phi^{++}(t_1,t_2)+\Phi^{+-}(t_1,t_2)+\Phi^{-+}(t_1,t_2)+\Phi^{--}(t_1,t_2)&=P_3\varphi\end{aligned}\right\}. \tag{7.14}$$

证明 显然, 由定理 4 的四个式子经过运算可得. **证毕**

第八章　复 Clifford 分析中的超正则函数

为区别起见本章用 $\mathcal{A}_n(R)$ 表示实 Clifford 代数空间，相当于以上各章中的 $\mathcal{A}$，用 $\mathcal{A}_n(C)$ 表示复 Clifford 代数空间，复 Cliffford 代数空间的基和实 Clifford 代数空间的相同，不同的是系数用复数. 实 Clifford 分析是研究从实向量空间 R^{n+1} 到 $\mathcal{A}_n(R)$ 的函数的数学分支. 复 Clifford 分析是研究从多复向量空间 C^{n+1} 到 $\mathcal{A}_n(C)$ 的函数的数学分支. 对复 Clifford 分析 F. Sommen, J. Ryan 等人做了许多工作，证明了复正则函数在位于复 Clifford 代数中的一个 Lie 群作用下的不变性，并且利用复调和函数去构造全纯函数. 黄沙 [25][26][27] 得到了复正则函数的充分必要条件和复正则函数与复调和函数的关系.

近年来 Clifford 分析中实超正则函数成为学者们研究的热门课题之一， W. Hengartner 和 H. Lentwiler 研究了 R^3 中的超正则函数，使 Clifford 分析中的函数理论有了进一步的发展， H.Leutwiler 在文 [28][29] 中研究了 H 型解，使超正则函数有了更加具体的表现形式.

本章借助于文献 [30] 受文献 [26] 的启发，得到了复 Clifford 分析中正则函数的等价条件，定义了复 Clifford 分析中的超正则函数，并研究了其若干性质，研究了复 Clifford 分析中的超正则函数与实 Clifford 分析中的函数的联系，并推广了文 [30] 和文 [26] 的结果.

§8.1　预备知识

复 Clifford 代数 $\mathcal{A}_n(C)$ 的基与实 Clifford 代数 $\mathcal{A}_n(R)$ 的一样，只是其系数用复数，即由以下形式的元素组成 $x=\sum\limits_A \alpha_A e_A,\quad \alpha_A\in C$.

复 Clifford 分析研究函数 $f: D\subset C^{n+1}\longrightarrow \mathcal{A}_n(C)$，$f(z)=\sum\limits_A f_A(z)e_A$，$f_A(z): D\subset C^{n+1}\longrightarrow C$(n+1 元复值函数)，自变量 $z=z_0+z_1e_1+\cdots+z_ne_n$.

复 Dirac 算子类似定义如下：设 $f: D\longrightarrow \mathcal{A}_n(C)$，则定义

$$\overline{\partial} f(z) = \sum_{j=0}^{n} e_j \frac{\partial f(z)}{\partial z_j} = \sum_{j=0}^{n} e_j \sum_A \frac{\partial f_A(z)}{\partial z_j} e_A,$$

$$\partial f(z) = e_0 \frac{\partial f(z)}{\partial z_0} - \sum_{j=1}^{n} e_j \frac{\partial f(z)}{\partial z_j},$$

其中 $\dfrac{\partial f_A(z)}{\partial z_j} = \dfrac{1}{2}\dfrac{\partial f_A(z)}{\partial x_j} - \dfrac{i}{2}\dfrac{\partial f_A(z)}{\partial y_j}$.

为方便计算，以下给出 $\mathcal{A}_n(R)$ ($\mathcal{A}_n(C)$) 中的一种对合运算和一种分解.

定义一种对合运算 $\prime : \mathcal{A}_n(R) \to \mathcal{A}_n(R)$ 如下,若 $x = \sum_A x_A e_A$，则 $x' = \sum_A x_A (-1)^{|A|} e_A$，其中 $|A|$ 表示集合 A 中元素的个数. 一种分解: 设 $x \in \mathcal{A}_n(R)$，$x = \sum_A x_A e_A$，分解 x 为 $x = b + c e_n$，b，$c \in \mathcal{A}_{n-1}(R)$, 定义两种映射

$$P,\ Q : \mathcal{A}_n(R) \to \mathcal{A}_{n-1}(R): \quad Px = b,\ Qx = c.$$

或

$$Px = \sum_{n \notin A} x_A e_A, \qquad Qx = \sum_{n \in A} x_A e_{A \setminus \{n\}}.$$

显然有

$$e_0' = e_0 = 1,\ e_A' = (-1)^{|A|} e_A,$$

$$(zw)' = z'w', \quad \forall\ z, w \in \mathcal{A}_n(R).$$

在实 Clifford 分析中引入修正的 Dirac 算子 $(Mf)(x) = (\overline{\partial} f)(x) + \frac{n-1}{x_n} Q' f(x)$, 其中 $Q'f = (Qf)'$. 若一个函数 f 在 $D \setminus \{x | x_n = 0\}$ 上满足 $Mf = 0$，则称 f 在 D 上为超正则函数.

若 f 在 D 上为超正则函数，又是向量值的 (就是其表达式中除 e_i, $i = 1, \cdots, n$ 的系数外都为 0), 则称 f 为 H 型解.

由正则函数到超正则函数并不是一种推广，谁也不包含谁，而是各自对应了有意义的方程.

§8.2 复正则函数

定义 8.1 令 D 是 C^{n+1} 中的一个区域，$f: D \longrightarrow \mathcal{A}_n(c)$ 是 D 上的一个全纯函数 ($f(z)$ 每个分支都是全纯函数), $f(z)=\sum\limits_{A} f_A(z)e_A$, $f_A(z)=u_A(z)+iv_A(z)$, 且对于每个 $z\in D$, 有

$$\overline{\partial} f(z)=\sum_{j=0}^{n} e_j \frac{\partial f(z)}{\partial z_j}=0, \tag{8.1}$$

则称 $f(z)$ 是 D 上的复正则函数，其中 $\dfrac{\partial f(z)}{\partial z_j}=\dfrac{1}{2}\dfrac{\partial f(z)}{\partial x_j}-\dfrac{i}{2}\dfrac{\partial f(z)}{\partial y_j}$ 如上所述.

定理 8.1 函数 $f(z)$ 是一个复值的 Clifford 函数，它的实部，虚部分别为实 Clifford 函数 $u(z), v(z)$, 则 $f(z)=u(z)+iv(z)$ 是复正则函数的充分必要条件为

$$\begin{aligned} \overline{\partial}_x u(z)+\overline{\partial}_y v(z) &= 0, \\ \overline{\partial}_y u(z)-\overline{\partial}_x v(z) &= 0. \end{aligned} \tag{8.2}$$

证明 记 $z_j=x_j+iy_j$, 由

$$\frac{\partial u(z)}{\partial z_j}=\frac{1}{2}\frac{\partial u(z)}{\partial x_j}-\frac{i}{2}\frac{\partial u(z)}{\partial y_j}, \qquad \frac{\partial v(z)}{\partial z_j}=\frac{1}{2}\frac{\partial v(z)}{\partial x_j}-\frac{i}{2}\frac{\partial v(z)}{\partial y_j},$$

有

$$\begin{aligned} \overline{\partial} f(z) &= \sum_{j=0}^{n} e_j \frac{\partial f(z)}{\partial z_j}=\sum_{j=0}^{n} e_j \frac{\partial[u(z)+iv(z)]}{\partial z_j} \\ &= \sum_{j=0}^{n} e_j \frac{\partial u(z)}{\partial z_j}+i\sum_{j=0}^{n} e_j \frac{\partial v(z)}{\partial z_j} \end{aligned}$$

$$= \frac{1}{2}\left(\sum_{j=o}^{n} e_j \frac{\partial u(z)}{\partial x_j} - i\sum_{j=0}^{n} e_j \frac{\partial u(z)}{\partial y_j} + i\sum_{j=0}^{n} e_j \frac{\partial v(z)}{\partial x_j} + \sum_{j=0}^{n} e_j \frac{\partial v(z)}{\partial y_j}\right)$$

$$= \frac{1}{2}\left(\sum_{j=0}^{n} e_j \frac{\partial u(z)}{\partial x_j} + \sum_{j=0}^{n} e_j \frac{\partial v(z)}{\partial y_j}\right) - \frac{i}{2}\left(\sum_{j=0}^{n} e_j \frac{\partial u(z)}{\partial y_j} - \sum_{j=0}^{n} e_j \frac{\partial v(z)}{\partial x_j}\right)$$

$$= \frac{1}{2}\left(\overline{\partial}_x u(z) + \overline{\partial}_y v(z)\right) - \frac{i}{2}\left(\overline{\partial}_y u(z) - \overline{\partial}_x v(z)\right),$$

则 $f(z)$ 为复正则函数的充分必要条件为 (8.2) 式成立. **证毕**.

由此我们得到了复正则函数的用其实部和虚部表示的充分必要条件, 从而把复 Clifford 函数与实 Clifford 函数联系起来, 实际上它是由复 Dirac 算子进一步推广得到的, 类似于单复变中的 *Cauchy − Riemann* 方程.

同实超正则函数一样, 我们可以用复正则函数的 Dirac 算子去定义复超正则函数, 可以得到复超正则函数的充分必要条件.

§8.3 复超正则函数

正如实向量 $x = x_0 + x_1e_1 + \cdots + x_ne_n$ 不是正则函数一样, 复向量 $z = z_0 + z_1e_1 + \cdots + z_ne_n$ 也不是复正则函数, 而是复超左正则函数 (以下简称复超正则函数). 以下我们引入复的修正的 Dirac 算子 M

$$(Mf)(z) = (\overline{\partial} f)(z) + \frac{n-1}{z_n}Q'f(z)$$
$$= (\overline{\partial} f)(z) + \frac{n-1}{|z_n|^2}\bar{z_n}Q'f(z).$$

由文献 [28] 知道,

$$(Mf)(e_J) = M(fe_J),$$

$$\frac{\partial Mf}{\partial z_j} = M\frac{\partial f}{\partial z_j}.$$

定义 8.2 令 D 是 C^{n+1} 的一个开子集，如果在子集 $D \setminus \{z|z_n = 0\}$ 上，函数 $f(z) \in A_n(C)$ 满足

$$Mf(z) = 0, \tag{8.3}$$

则函数 $f(z)$ 叫做 D 上的复超正则函数.

推论 8.1 令 D 是 C^{n+1} 的一个开子集，$f : D \longrightarrow A_{n-1}(z)$ 是复正则函数，则 $f(z)$ 是复超正则函数.

记算子

$$\begin{aligned} M_x &= \tfrac{1}{2}\overline{\partial}_x + \tfrac{n-1}{|z_n|^2}x_n Q', \\ M_y &= \tfrac{1}{2}\overline{\partial}_y + \tfrac{n-1}{|z_n|^2}y_n Q'. \end{aligned} \tag{8.4}$$

定理 8.2 令 D 是 C^{n+1} 中的一个开子集，$f : D \longrightarrow A_n(C)$ 是复超正则函数，则

$\dfrac{\partial f(z)}{\partial z_j}, \dfrac{\partial f(z)}{\partial x_j}, \dfrac{\partial f(z)}{\partial y_j}$ 都是复超正则函数.

证明 因为 $f(z)$ 是复超正则函数，故 $Mf(z) = 0$.

所以

$$M\frac{\partial f(z)}{\partial x_j} = \frac{\partial Mf(z)}{\partial x_j} = 0,$$

$$M\frac{\partial f(z)}{\partial y_j} = \frac{\partial Mf(z)}{\partial y_j} = 0,$$

$$M\frac{\partial f(z)}{\partial z_j} = \frac{\partial Mf(z)}{\partial z_j} = 0.$$

所以 $\dfrac{\partial f(z)}{\partial z_j}, \dfrac{\partial f(z)}{\partial x_j}, \dfrac{\partial f(z)}{\partial y_j}$ 都是复超正则函数. **证毕** .

定理 8.3 函数 $f : D \longrightarrow A_n(C)$ 是复超正则函数的充分必要条件为

$$\begin{aligned} M_x u(z) + M_y v(z) &= 0, \\ M_y u(z) - M_x v(z) &= 0, \end{aligned} \tag{8.5}$$

这里 M_x, M_y 由 (8.4) 式给出.

证明

$$\begin{aligned} Mf(z) &= \overline{\partial} f(z) + \frac{n-1}{|z_n|^2}\bar{z}_n Q' f(z) \\ &= \frac{1}{2}\overline{\partial}_x u(z) + \frac{1}{2}\overline{\partial}_y v(z) - \frac{i}{2}\overline{\partial}_y u(z) - \frac{i}{2}\overline{\partial}_x v(z) \\ &\quad + \frac{n-1}{|z_n|^2}(x_n - iy_n)\left(Q'u(z) + iQ'v(z)\right) \\ &= \frac{1}{2}\overline{\partial}_x u(z) + \frac{1}{2}\overline{\partial}_y v(z) - \frac{i}{2}\overline{\partial}_y u(z) - \frac{i}{2}\overline{\partial}_x v(z) + \frac{n-1}{|z_n|^2}x_n Q'u(z) \\ &\quad + \frac{n-1}{|z_n|^2}y_n Q'v(z) - i\frac{n-1}{|z_n|^2}y_n Q'u(z) + i\frac{n-1}{|z_n|^2}x_n Q'v(z) \\ &= \frac{1}{2}\overline{\partial}_x u(z) + \frac{n-1}{|z_n|^2}x_n Q'u(z) + \frac{1}{2}\overline{\partial}_y v(z) + \frac{n-1}{|z_n|^2}y_n Q'v(z) \\ &\quad - i\left[\frac{1}{2}\overline{\partial}_y u(z) + \frac{n-1}{|z_n|^2}y_n Q'u(z) - \left(\frac{1}{2}\overline{\partial}_x v(z) + \frac{n-1}{|z_n|^2}x_n Q'v(z)\right)\right] \\ &= \left(M_x u(z) + M_y v(z)\right) - i\left(M_y u(z) - M_x v(z)\right), \end{aligned}$$

所以 $f(z)$ 为复超正则函数的充分必要条件为 (8.5) 式成立. **证毕** .

定义 8.3 如果 $Mf(z)=0$, 且 $f(z)=\sum_{j=0}^{n}f_j(z)e_j$, 则 $f(z)$ 叫做复 H 型解.

易知 $z=\sum_{j=0}^{n}z_je_j$ 是一个复 H 型解.

注:

(1) H 型解全体是复超正则函数的非空真子集.

例 $f:D\subset C^n\longrightarrow\mathcal{A}_n(C)$, $f(z)=2z_1z_2e_1e_3+({z_1}^2-{z_2}^2)e_2e_3$, 则 f(z) 为复超正则函数, 但不是 H 型解.

(2) 复超正则函数乘复超正则函数不一定是复超正则函数; H 型解乘 H 型解不一定是 H 型解, 甚至不一定是复超正则函数.

例 $f_1(z)=(2-n)z_0+z_ne_n$, $f_2(z)=(2-n)z_1e_1+z_ne_n$ 为两个 H 型解, 其乘积不是 H 型解, 也不是复超正则函数.

自然要问什么时候复超正则函数乘复超正则函数仍是复超正则函数? H 型解乘 H 型解是 H 型解? 以下定理部分地回答了这些问题.

定理 8.4 令 D 是 C^{n+1} 中的一个开子集, $f:D\longrightarrow\mathcal{A}_n(C)$ 是复超正则函数, 则乘积 $F(z)=f(z)z$ 是复超正则函数的充分必要条件为 $f(z)$ 是一个复 H 型解.

为了证明定理 8.4, 我们首先给出几个预备引理.

引理 8.1 (1) $x\in\mathcal{A}_n(R)$ 是一个向量值的 Clifford 数, 即 $x=\sum_{i=0}^{n}x_ie_i$, 当且仅当 $\sum_{j=0}^{n}e_jxe_j=(1-n)x'$.

(2) $z\in\mathcal{A}_n(C)$ 是一个复向量值的 Clifford 数当且仅当 $\sum_{j=0}^{n}e_jze_j=(1-n)z'$.

证明 若 $x=\sum_{i=0}^{n}x_ie_i$, 则

$$\sum_{j=0}^{n} e_j x e_j = \sum_{j=0}^{n} e_j (\sum_{i=0}^{n} x_i e_i) e_j = \sum_{j=0}^{n}\sum_{i=0}^{n} x_i e_j e_i e_j = (1-n)x'.$$

反之, 设 $x = \sum_A x_A e_A$ 满足 $\sum_{j=0}^{n} e_j x e_j = (1-n)x'$, 要证对任意的 $A = \{i_1, i_2, \cdots, i_k | |A| = k \geq 2\}$ 有 $x_A = 0$. 由题设知

$$\sum_{j=0}^{n} e_j (\sum_A x_A e_A) e_j = (1-n) \sum_A x_A (-1)^{|A|} e_A$$

上式左端为 $\sum_{j=0}^{n} e_j (\sum_A x_A e_A) e_j = \sum_{j,A} x_A e_j e_A e_j = \sum_{j,A} x_A (\pm 1) e_A$, 其中 '+', '−' 交替出现, 所以当 $|A| \geq 2$ 时, 系数的绝对值一定小于 $(n-1)|x_A|$; 又知 $(1-n)\sum_A x_A(-1)^{|A|} e_A$ 对 $A = \{i_1, i_2, \cdots, i_k | k \geq 2\}$ 有 e_A 系数的绝对值等于 $(n-1)|x_A|$, 由以上等式可得此时一定有 $x_A = 0$.(1) 证毕. (2) 由 (1) 易知. **证毕**.

引理 8.2 设 $x, y \in \mathcal{A}_n(R)$ (或 $\mathcal{A}_n(c)$), P, Q 为投影算子, 则

$$(Px)' = P(x'), \quad (Ox)' = -Q(x'),$$

$$P(xy) = (Px)Py + Q(x)Q(y'), \quad Q(xy) = (Px)Qy + Q(y)P(y').$$

证明 由投影算子 P, Q 的定义知

$$x = Px + (Qx)e_n; \quad y = Py + (Qy)e_n,$$

易知前两式成立. 又因为

$$\begin{aligned} xy &= (Px + (Qx)e_n)(Py + (Qy)e_n) \\ &= (Px)(Py) + (Px)(Qy)e_n + (Qx)e_n Py + (Qx)e_n(Qy)e_n \\ &= (Px)(Py) + (Qx)(Qy' + [(Px)Qy + (Qx)(Py')]e_n, \end{aligned}$$

可知后两式也成立. **证毕**.

引理 8.3 设 $D \subset R^{n+1}$, $f, g : D \longrightarrow \mathcal{A}_n(R)$, 关于每个 x_i 的导数都存在, 则有微分公式

$$\overline{\partial}(fg) = (\overline{\partial} f)g + \sum_{j=0}^{n} e_j f \frac{\partial g}{\partial x_j}.$$

证明 设 $f = \sum\limits_{A} x_A e_A$, $g = \sum\limits_{B} x_B e_B$, 则

$$\begin{aligned}
\overline{\partial}(fg) &= \sum_{j=0}^{n} e_j \tfrac{\partial (fg)}{\partial x_j} \\
&= \sum_{j} e_j \sum_{A \cdot B} \tfrac{\partial (x_A y_B)}{\partial x_j} e_A e_B \\
&= \sum_{j=0}^{n} e_j \sum_{A \cdot B} e_A e_B [\tfrac{\partial x_A}{\partial x_j} y_B + x_A \tfrac{\partial y_B}{\partial x_j}] \\
&= \sum_{j=0}^{n} e_j \sum_{A} \tfrac{\partial x_A}{\partial x_j} e_A \sum_{B} y_B e_B + \sum_{j} e_j \sum_{A} x_A e_A \sum_{B} \tfrac{\partial y_B}{\partial x_j} e_B \\
&= (\overline{\partial} f)g + \sum_{j=0}^{n} e_j f \tfrac{\partial g}{\partial x_j}.
\end{aligned}$$

证毕.

定理 8.4 **的证明**

对于 $z \in D \subset C^{n+1}$, 有

$$\begin{aligned} z &= z_0 + z_1e_1 + \cdots + z_ne_n \\ &= (x_0 + iy_0) + (x_1 + iy_1)e_1 + \cdots + (x_n + iy_n)e_n \\ &= (x_0 + x_1e_1 + \cdots + x_ne_n) + i(y_0 + y_1e_1 + \cdots + y_ne_n) \\ &= x + iy, \end{aligned}$$

$$\begin{aligned} f(z) &= \sum_J f_J(z)e_J \\ &= \sum_J u_J(z)e_J + i\sum_J v_J(z)e_J \\ &= u(z) + iv(z), \end{aligned}$$

因此

$$\begin{aligned} F(z) &= f(z)z \\ &= (u + iv)(x + iy) \\ &= (ux - vy) + i(vx + uy) \\ &= U(z) + iV(z), \end{aligned}$$

这里

$$U(Z) = u(z)x - v(z)y, \quad V(Z) = v(z)x + u(z)y.$$

由引理 8.2 有

$$\begin{aligned}
Q'(ux) &= [(Pu)Qx + (Qu)P(x')]' \\
&= [(Pu)x_n + (Qu)P(x')]' \\
&= P'(u)x_n + Q'(u)P(x) \\
&= P'(u)x_n + Q'(u)(x - x_n e_n) \\
&= P'(u)x_n + Q'(u)x - Q'(u)x_n e_n, \\
Q'(vy) &= P'(v)y_n + Q'(v)y - Q'(v)y_n e_n.
\end{aligned}$$

又由引理 8.3 有

$$\begin{aligned}
M_x(ux) &= \tfrac{1}{2}\overline{\partial}_x(ux) + \tfrac{n-1}{|z_n|^2}x_n Q'(ux) \\
&= (\tfrac{1}{2}\overline{\partial}_x u)x + \tfrac{1}{2}\sum_{j=0}^{n} e_j u e_j \\
&\quad + \tfrac{n-1}{|z_n|^2}[x_n^2 P'(u) - x_n^2 Q'(u)e_n] + \tfrac{n-1}{|z_n|^2}x_n Q'(u)x \\
&= (\tfrac{1}{2}\overline{\partial}_x u)x + \tfrac{n-1}{|z_n|^2}x_n Q'(u)x + \tfrac{1}{2}\sum_{j=0}^{n} e_j u e_j \\
&\quad + \tfrac{n-1}{|z_n|^2}[x_n^2 P'(u) - x_n^2 Q'(u)e_n]
\end{aligned}$$

$$
\begin{aligned}
&= [\tfrac{1}{2}\overline{\partial}_x + \tfrac{n-1}{|z_n|^2}x_n Q'](u)x + \tfrac{1}{2}\sum_{j=0}^{n} e_j u e_j \\
&\quad + \tfrac{n-1}{|z_n|^2}[{x_n}^2 P'(u) - {x_n}^2 Q'(u)e_n] \\
&= (M_x u)x + \tfrac{1}{2}\sum_{j=0}^{n} e_j u e_j + \tfrac{n-1}{|z_n|^2}[{x_n}^2 P'(u) - {x_n}^2 Q'(u)e_n],
\end{aligned}
$$

$$
\begin{aligned}
M_x(vy) &= \tfrac{1}{2}\overline{\partial}_x(vy) + \tfrac{n-1}{|z_n|^2}x_n Q'(vy) \\
&= (\tfrac{1}{2}\overline{\partial}_x v)y + \tfrac{n-1}{|z_n|^2}x_n\,[P'(v)y_n + Q'(v)y - Q'(v)y_n e_n] \\
&= (\tfrac{1}{2}\overline{\partial}_x + \tfrac{n-1}{|z_n|^2}x_n Q')(v)y \\
&\quad + \tfrac{n-1}{|z_n|^2}[x_n P'(v)y_n - x_n Q'(v)y_n e_n] \\
&= (M_x v)y + \tfrac{n-1}{|z_n|^2}[x_n P'(v)y_n - x_n Q'(v)y_n e_n].
\end{aligned}
$$

同理，有

$$
\begin{aligned}
M_y(vx) &= (M_y v)x + \tfrac{n-1}{|z_n|^2}[y_n P'(v)x_n - y_n Q'(v)x_n e_n], \\
M_y(uy) &= (M_y u)y + \tfrac{1}{2}\sum_{j=0}^{n} e_j u e_j + \tfrac{n-1}{|z_n|^2}[y_n^2 P'(u) - Q'(u)e_n y_n^2],
\end{aligned}
$$

所以

$$
M_x U + M_y V = M_x(ux - vy) + M_y(vx + uy)
$$

$$
= M_x(ux) - M_x(vy) + M_y(vx) + M_y(uy)
$$

$$= (M_x u)x + \frac{1}{2}\sum_{j=0}^{n} e_j u e_j + \frac{n-1}{|z_n|^2}[{x_n}^2 P'(u) - {x_n}^2 Q'(u)e_n]$$

$$-(M_x v)y - \frac{n-1}{|z_n|^2}[x_n P'(v)y_n - x_n Q'(v)y_n e_n]$$

$$+(M_y v)x + \frac{n-1}{|z_n|^2}[y_n P'(v)x_n - y_n Q'(v)x_n e_n]$$

$$+(M_y u)y + \frac{1}{2}\sum_{j=0}^{n} e_j u e_j + \frac{n-1}{|z_n|^2}[{y_n}^2 P'(u) - {y_n}^2 Q'(u)e_n]$$

$$= (M_x u + M_y v)x + (M_y u - M_x v)y + \sum_{j=0}^{n} e_j u e_j$$

$$+\frac{n-1}{|z_n|^2}[|z_n|^2 P'(u) - |z_n|^2 Q'(u)e_n].$$

因为 $f(z)$ 是复超正则函数，根据定理 8.3，由 (8.4) 式，有

$$
\begin{aligned}
& M_x U + M_y V \\
= \ & (n-1)[P'(u) - Q'(u)e_n] + \sum_{j=0}^{n} e_j u e_j \\
= \ & (n-1)[P(u') + Q(u')e_n] + \sum_{j=0}^{n} e_j u e_j \\
= \ & (n-1)u' + \sum_{j=0}^{n} e_j u e_j,
\end{aligned}
$$

同理有

$$M_y U - M_x V = M_y(ux - vy) - M_x(vx + uy)$$

$$= M_y(ux) - M_y(vy) - M_x(vx) - M_x(uy)$$

$$= (n-1)v' + \sum_{j=0}^{n} e_j v e_j,$$

于是由定理 8.3, 引理 8.1 知

$F(z)$是复超正则函数.

$$\Longleftrightarrow \begin{cases} M_x U(z) + M_y V(z) &= 0 \\ M_y U(z) - M_x V(z) &= 0 \end{cases}$$

$$\Longleftrightarrow \begin{cases} (n-1)u' + \sum\limits_{j=0}^{n} e_j u e_j &= 0 \\ (n-1)v' + \sum\limits_{j=0}^{n} e_j v e_j &= 0 \end{cases}$$

$\Longleftrightarrow$ u, v分别是实向量值.

$\Longleftrightarrow$ $f(z)$是一个复向量值，又已知$f(z)$是复超正则函数.

$\Longleftrightarrow$ $f(z)$是一个复H型解.

证毕.

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