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A planar mechanical library in the AMESim simulation software. Part I: Formulation of dynamics equations

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Abstract

This paper presents the mathematical developments of a planar mechanical library implemented in the AMESim simulation tool. Body and joint components are the basic components of this library. Due to the library philosophy requirements, the mathematical models of the components have required a generic vector calculus based formulation of the constraint equations. This formulation uses a set of dependent generalized coordinates. The dynamics equations are obtained from the application of Jourdain's principle combined with the Lagrange multiplier method. The body component mathematical models consist of differential equations in terms of the dependent generalized coordinates. The joint component mathematical models are based on the Baumgarte stabilization schemes applied to the geometrical, kinematic and acceleration constraint equations. The Lagrange multipliers are the implicit solution of these Baumgarte stabilization schemes. The first main contribution of this paper is the expression of geometrical constraints in terms of vectors and their exploitation in this form. The second important contribution is the adaptation of existing formulations to the AMESim philosophy.

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1. Introduction

This paper, organized in two parts, presents a new library for the simulation tool AMESim [2]. The first part is dedicated to the theoretical developments of the library. The second part shows the composition of the library as it was primarily implemented in AMESim and illustrates it with an application example of a seven-body mechanism. This library proposes components belonging to the planar mechanical domain. The objective with this library was not to compete with multibody system software tools that are better adapted to this domain. The objective was more to enlarge the range of industrial applications capable of being treated by AMESim. From a theoretical point of view the challenge of implementing this library was to fit existing mechanical formulations to the inherent requirements of AMESim philosophy. The solution has been found by adapting the dynamic equations expressed from Jourdain's principle and the Lagrange multiplier method together with Baumgarte's stabilization. Also a generic feature of the formulation has been researched over the library components (bodies and joints) and one key contribution of this paper is concerned with this generic feature. Basically the formulation consists of expressing the geometric constraints associated with joints in terms of vectors and carrying out the developments of this form. The result is the set up, for kinematic and acceleration constraints, of a unique expression that fits every joint presented in the library.

The generic feature of the formulation proposed thus enables the derivation of joint contraints to be systematized. One can then imagine a new joint with its corresponding vector constraint and derive straightforwardly the corresponding mathematical model by applying the proposed formulation. Also, in the context of predefined component models, the given formulation clearly shows the frontiers of the different mathematical models in terms of inputs and outputs. Therefore it also helps to define in which models output equations must be implemented. Also, the formulation proposed intrinsically enables closed loop structures to be dealt with.

AMESim (for Advanced Modeling Environment for performing Simulations) is organized in component libraries. The components, represented by symbolically technologically suggested icons, can be interconnected exactly like the system under study. AMESim was first applied to electrohydraulic engineering systems with simple one-dimensional mechanical systems (like inertia, springs, and dampers in translation or in rotation). It recently opened its libraries to a variety of other component domains. One can now carry out modeling, analysis and simulation for systems consisting of pneumatic, powertrain, hydraulic resistance, thermal, electromagnetic and cooling components for instance. The restriction to only one-dimensional motion for the mechanical components has motivated the development of a two-dimensional mechanical library.

Section 2 presents an overview of some multibody codes and object-oriented tools, as well as the environmental requirements of AMESim. These requirements have some implications on how the 2D library is built. Section 3 details the theoretical developments that enabled the mathematical models of the library components to be set up. Section 4 concludes this first part.

2. Constraints of AMESim library philosophy

After a brief overview of multibody code principles and some object-oriented tools, a presentation of AMESim requirements is given.

Concerning multibody codes a state of the art is given by [23]. Details are not reproduced here and readers are referred to this book for a more profound presentation. Although more than a decade has passed and certain tools are no longer developed and others have changed, this state of the art book gives a good idea of the main principles that can be used as a basis for multibody codes. Also this overview enables the library proposed to be positioned with respect to these codes. There are different approaches for writing dynamic equations. The approaches most represented in multibody codes are, the Newton–Euler equations applied to each body, the Newton–Euler equations applied to sets of bodies, Lagrange's equations and Kane's equations [13,14]. The variables, in whose terms the dynamic equations are written, are either absolute coordinates or relative coordinates. Also supplementary methods are used for reducing the index of the Differential–Algebraic Equations. The principal ones are the coordinate partitioning method, the projection matrix method, the Baumgarte stabilization and the penalty formulation [9]. The first two methods aim at working with a set of independent generalized coordinates while the Baumgarte stabilization enables the constraints, together with the differential equations, to be handled and the penalty formulation increases the differential system order by introducing extra dynamics into the model.

In the domain of the object-oriented tools to which AMESim may be attached, certain enable multibody systems to be treated with a different approach to the modelling. For instance Dymola [21] is, like AMESim, based on well-identified technological components in a pluridisciplinary context but it sets up the mathematical model in a different way. Basically each component model consists of equations not oriented in terms of variable assignments nor organized a priori. Then, at the component connection stage, all the mathematical models are gathered in an implicit form and the compilation carries out the variable assignments and the organization of the equations in a consistent manner. Thus the order of the whole model is globally reduced and a number of constraints are a priori symbolically eliminated. Likewise, tools based on bond graph (e.g. 20Sim [1] or MS1 [18]) can deal with multibody systems in a pluridisciplinary context (e.g. [4,7]). The essential feature of bond graph language is its ability to describe the energy topology of a model at an acausal level. This enables all the model variables to be globally assigned and all the equations to be globally organized. This also eliminates superfluous dependencies of the multibody models.

It is now important to show the key features of AMESim to justify how the planar mechanical library has been implemented. Its feature oriented towards engineering systems and its user friendliness make AMESim work with well-identified technological components, symbolically manipulated by means of icons. These icons are interconnected, one to the other and identically to the engineering system architecture under study. Fig. 1 shows an example of a door locking system using a permanent magnet modelled in AMESim. The icons displayed here belong to the magnetic, mechanical and signal libraries. This simple example shows the coupling between mechanical and magnetic domains where one circuit, fed by a permanent magnet (right-hand side magnetic circuit), is forced to move with respect to another passive circuit (left-hand side circuit). The main components consist of a permanent magnet (rectangle with a compass needle inside), three magnetic circuit parts characterized by a certain reluctance (rectangles with 'square' ports with a diagonal cross inside), two variable air-gaps (vertical twin rectangles), two mechanical nodes (both sides of the air-gap components), a signal generator with a signal-to-displacement converter (in the centre of the right-hand side circuit), and a component for the set of the magnetic medium characteristics (B–H diagram in a circle). Each component can be associated with one model from a set of component compatible mathematical models. As soon as the model has been chosen the component conserves this mathematical model.

Contrary to acausal tools, AMESim works with component models that have equations both a priori oriented in terms of variable assignments and organized. This feature requires implementing new models in a predefined calculus scheme. Also the mathematical formulation of a component model has to be organized in order to fit into other potential component connections. So each component associated with a mathematical model has a predetermined set of input and output variables. It can thus be considered as a causal model. The connection of the components enables the exchange of those variables on the way out a component for those variables that are calculated by its mathematical model (outputs) and, the exchange of those variables on the way in a component for those variables that are calculated by a connected component mathematical model (inputs). This causal feature of AMESim philosophy is the main constraint when implementing new components. This differs

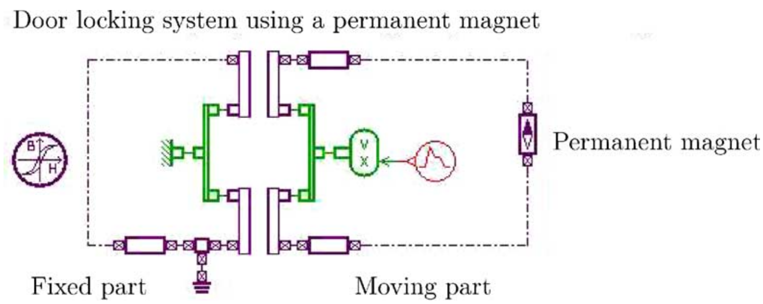

Fig. 1. Example of an AMESim model representation.

from other object-oriented tools, based on acausal component models or acausal phenomenon models, like Dymola, or tools with a bond graph input (e.g. 20-Sim or MS1).

Fig. 2 gives an example of two components in the mechanical (a mass in translation) and the hydraulic (a two way hydraulic pump) domains respectively. The connecting ports of the components show the variables exchanged by them and especially the outputs ('exiting' arrows) passed to the connected components and the inputs ('entering' arrows) received from the connected components. These connecting ports are intimately associated with power ports since two of the variables exchanged at these ports are power variables.

Fig. 2 examples illustrate two key features of a library oriented simulation tool. The first one is the domain port concept. It shows how AMESim can deal with pluridisciplinary systems. The second feature is the connecting port constraints. Since one component mathematical model requires given inputs to then calculate its state and its outputs, not all combinations of the component connections are allowed. For instance the Fig. 2 examples cannot be connected one to the other by any port. However a mass component may be connected to a spring component or a damper component.

Another key feature of a library oriented simulation software tool is the modularity concept. This often results in symmetrical components with respect to their connecting port. This symmetry property, though not generalized to all components in AMESim, has been adopted for the planar mechanical library. The reason will appear obvious when components of this library are presented.

In the context of planar mechanisms and rigid bodies the library is not restricted to any mechanical domain application. The library also accepts closed loop structures. Although relative coordinates are generally more efficient for dynamic equation formulation, AMESim philosophy requires the use of absolute coordinates. The absolute coordinates of the mass center have been chosen for each body. Nevertheless the planar feature of the library does not require any specific variables for the body orientation. Thus the absolute angular position has been chosen for each body as well. Once again, due to AMESim philosophy, the equations of the components cannot be globally reorganized when the components are connected. This forbids the use of the coordinate partitioning method or the projection method to decrease the index of the Differential–Algebraic Equation systems. For this reason the Baumgarte stabilization has also been used in the library.

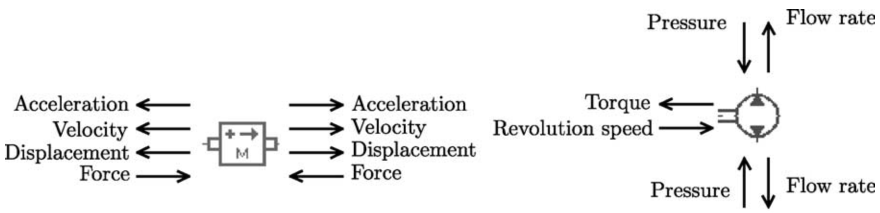

Fig. 2. Example of two AMESim components.

3. Theoretical developments of the library components

As has been explained in the previous section the library must be organized in well-identified technological components. It has been decided to base the planar mechanical library on a body component and on joint components. The body component is associated with a supposed rigid material item of a mechanism. Its behavior is essentially governed by its kinetic state. The joint components are associated with the abstract items that represent the attachment of bodies in a mechanism. They are supposed to be ideal and their mathematical model is based on the constraints that they impose on the connected bodies.

3.1. Body component mathematical model

The mathematical model of the body component is based on Jourdain's Principle formulation (e.g. [5,23])[1]:

$$A^* = P^* \tag{1}$$

where A^* is the virtual power developed by the acceleration quantities and P^* the virtual power developed by the actions on the body.

In the library philosophy there is no a priori privileged candidate for the role of the generalized coordinates. For a planar motion, the generalized coordinates, which have been chosen, are the absolute mass center coordinates projected onto the absolute frame of reference (x_{G_i}, y_{G_i}) and the absolute angular position θ_i (Fig. 3). This choice enables the more general case of a body motion to be dealt with. The body motion restriction will be determined by the joint constraints, as shown later.

With this choice of generalized coordinates $(x_{G_i}, y_{G_i}, \theta_i)$ Eq. (1) members may now be written

$$\begin{aligned} A^* &= A_x\dot{x}^*_{G_i} + A_y\dot{y}^*_{G_i} + A_\theta\dot{\theta}^*_i \\ P^* &= Q_x\dot{x}^*_{G_i} + (Q_y - m_i g)\dot{y}^*_{G_i} + Q_\theta\dot{\theta}^*_i \end{aligned} \tag{2}$$

with m_i the body mass and g the gravity acceleration. We consider here $\vec{y}_0$ as the ascendant vertical axis. The star superscript indicates virtual quantities. The coefficients of the virtual velocities in A^* are derived from the kinetic coenergy (e.g. [6]) of a body by the equation:

$$A_q = \frac{\mathrm{d}\partial T}{\mathrm{d}t\,\partial\dot{q}} - \frac{\partial T}{\partial q} \quad \text{with } q \text{ a generalized coordinate} \tag{3}$$

[1] A nomenclature is given in Appendix A.

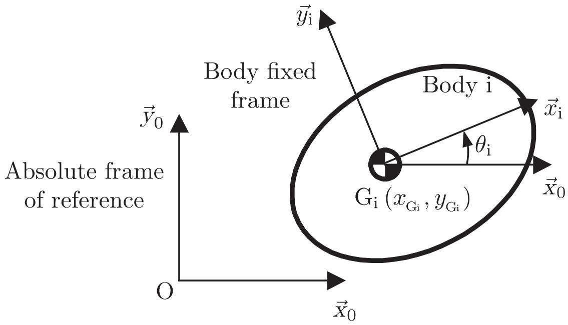

Fig. 3. Schema of a body in planar motion.

Applied to Fig. 3 body in planar motion these quantities are written simply:

$$\begin{aligned} A_x &= m_i\ddot{x}_{G_i} \\ A_y &= m_i\ddot{y}_{G_i} \quad \text{with } I_i \text{ the body moment of inertia around } (G_i, \vec{z}_0) \\ A_\theta &= I_i\ddot{\theta}_i \end{aligned} \tag{4}$$

Q_x, Q_y, and Q_θ are the generalized forces including the constraint actions resulting from the fact that x_{G_i}, y_{G_i}, and θ_i are not necessarily independent after the connection of a body component to a joint component. From Eq. (1) and by taking a compatible virtual transformation with the joints as they exist at time t, we can now write the three identities that constitute the formulation basis for the body components. These three identities are

$$\begin{aligned} m_i\ddot{x}_{G_i} &= Q_x \\ m_i\ddot{y}_{G_i} &= Q_y - m_i g \\ I_i\ddot{\theta}_i &= Q_\theta \end{aligned} \tag{5}$$

This formulation requires that the expression of the three generalized forces Q_x, Q_y, and Q_θ be further developed in order to fit any potential connected joint component.

First let us inspect the case of a body with only one connecting port at a point M. Let us also consider simply a given action on the body characterized by a wrench about point M (e.g. [17]):

$$\{\mathscr{W}\} : \begin{cases} \vec{F} = F_x\vec{x}_0 + F_y\vec{y}_0 & \text{force} \\ \vec{M}(M) = C_z\vec{z}_0 & \text{torque about point } M \end{cases} \tag{6}$$

The virtual power developed by this action is

$$P^* = \vec{F} \cdot \vec{V}^{0^*}(M) + \vec{M}(M) \cdot \vec{\Omega}_i^{0^*} \tag{7}$$

where $\vec{V}^{0^*}(M)$ is the virtual absolute velocity of point M and $\vec{\Omega}_i^{0^*}$ is the virtual absolute angular velocity of the body. The velocity transport (e.g. [11]) enables Eq. (7) to be written as:

$$\begin{aligned} P^* &= \vec{F} \cdot \vec{V}^{0^*}(G_i) + \vec{F} \cdot \left(\vec{\Omega}_i^{0^*} \times \overrightarrow{G_iM}\right) + \vec{M}(M) \cdot \vec{\Omega}_i^{0^*} \\ &= \vec{F} \cdot (\dot{x}_{G_i}^* \vec{x}_0 + \dot{y}_{G_i}^* \vec{y}_0) + \left(\vec{M}(M) + \overrightarrow{G_iM} \times \vec{F}\right) \cdot \dot{\theta}_i^* \vec{z}_0 \end{aligned} \tag{8}$$

From Eq. (8) we can clearly identify the generalized forces used in the dynamic formulation of a body component:

$$\begin{aligned} Q_x &= \vec{F} \cdot \vec{x}_0 = F_x \\ Q_y &= \vec{F} \cdot \vec{y}_0 = F_y \\ Q_\theta &= \left(\vec{M}(M) + \overrightarrow{G_iM} \times \vec{F}\right) \cdot \vec{z}_0 = C_z + \vec{F} \cdot (\vec{z}_0 \times \overrightarrow{G_iM}) \end{aligned} \tag{9}$$

Since $\overrightarrow{G_iM}$ is a characteristic vector of the body, the variables F_x, F_y, and C_z, characterizing the given force at point M, are the only variables passed to the body at the connecting port. The variables Q_x, Q_y, and Q_θ are calculated in the body component model on Eq. (9) basis.

It is shown in the next section that the equation formulation for the generalized forces (Eq. (9)) applies for any type of joint component connected to a body component. The expressions of F_x, F_y, and C_z vary with the type of the connected joint but are calculated in the joint component.

3.2. Joint component mathematical model

First a general formulation is given for the joint component mathematical model. It is then illustrated in the example of a translational joint.

Let us consider this time two bodies connected by a joint. By the only fact that both bodies are connected (a joint component between two body components) their generalized coordinates (x_{G_i}, y_{G_i} and θ_i for body i and x_{G_j}, y_{G_j} and θ_j for body j) are no longer independent. In the library philosophy the constraint equations are expressed in the joint component, which in turn furnishes the constraint actions to the body components. These constraint actions correspond to the variables F_x, F_y, and C_z previously presented and passed to each body component. The general expressions of these variables are now determined.

The joints considered in the planar mechanical library generate only geometrical constraints. These constraints may be expressed in a general way in an implicit form by Eq. (10) (e.g. [15]).

$$g_k(q_1, \ldots, q_n) = 0 \quad \text{for } k = 1 \text{ to } m \tag{10}$$

with n the number of generalized coordinates involved in the constraints and m the constraint number.

It is supposed here that the constraints are scleronomic [16], which means that time does not explicitly appear in the constraint equations. At the kinematic level these equations become

$$\dot{g}_k(q_1,\ldots,q_n,\dot{q}_1,\ldots,\dot{q}_n) = \frac{\partial g_k}{\partial q_i}\dot{q}_i = 0$$

for $k = 1$ to m and with the Einstein implicit summation convention on repeated subscript i (11)

In the case of bodies connected by a joint the generalized coordinates are

$$\underline{q}^{\mathrm{T}} = \begin{bmatrix} x_{G_i} & y_{G_i} & \theta_i & x_{G_j} & y_{G_j} & \theta_j \end{bmatrix} \quad (12)$$

The library philosophy imposes working with a set of dependent coordinates. A Lagrange multiplier is then associated with each constraint equation of the joint between both bodies. Let λ_k (k = 1 to m) be these Lagrange multipliers. We now develop the expressions of the resulting constraint terms F_{x_i}, F_{y_i}, and C_{z_i} on the body i side and F_{x_j}, F_{y_j}, and C_{z_j}, on the body j side in terms of the Lagrange multipliers λ_k (k = 1 to m) and the joint geometry.

Let us first consider the Fig. 4 schematic representation of two bodies in a general planar motion (note that in the context of a planar motion, $\vec{z}_i = \vec{z}_j = \vec{z}_0$). The joint is characterized by the geometric axes $(M, \vec{x}_{il})$ and $(N, \vec{x}_{jl})$ respectively on bodies i and j (no peculiarity is shown at this point in order to keep the generality of the development). These two axes are defined by their relative positions in the corresponding body frames (relative coordinates $(x_{\mathrm{M}}, y_{\mathrm{M}})$ and $(x_{\mathrm{N}}, y_{\mathrm{N}})$ with respect to body frames, respectively, for points M and N, and angular relative positions θ_{il} and θ_{jl} for the vector axes $\vec{x}_{il}$ and $\vec{x}_{jl}$).

The geometrical joints considered between bodies i and j may also be expressed by the following equations (this conjecture is at least verified for joints considered in the library)[2]:

$$f_k\left(\overrightarrow{OM}, \overrightarrow{ON}, \vec{x}_{il}, \vec{y}_{il}, \vec{x}_{jl}, \vec{y}_{jl}\right) = 0 \quad \text{for } k = 1 \text{ to } m \quad (13)$$

Next the kinematic equations corresponding to these geometrical constraints are obtained by differentiation with respect to time and in the same reference frame. Let the absolute frame of reference be the frame of differentiation. The kinematic constraints are (see the appendix for the calculus details and the notations used)

$$\begin{aligned}
&\vec{f}_{k,\overrightarrow{OM}} \cdot \left(\dot{x}_{G_i}\vec{x}_0 + \dot{y}_{G_i}\vec{y}_0\right) + \left(\overrightarrow{G_iM} \times \vec{f}_{k,\overrightarrow{OM}} - \vec{f}_{k,\vec{x}_{il}} \times \vec{x}_{il} - \vec{f}_{k,\vec{y}_{il}} \times \vec{y}_{il}\right) \cdot \dot{\theta}_i\vec{z}_0 \\
&\quad + \vec{f}_{k,\overrightarrow{ON}} \cdot \left(\dot{x}_{G_j}\vec{x}_0 + \dot{y}_{G_j}\vec{y}_0\right) + \left(\overrightarrow{G_jN} \times \vec{f}_{k,\overrightarrow{ON}} - \vec{f}_{k,\vec{x}_{jl}} \times \vec{x}_{jl} - \vec{f}_{k,\vec{y}_{jl}} \times \vec{y}_{jl}\right) \cdot \dot{\theta}_j\vec{z}_0 = 0
\end{aligned}$$

for $k = 1$ to m (14)

[2] Even if Eqs. (10) and (13) correspond to the same constraint equations, there is a fundamental distinction between their expressions. In fact g_k may be defined as a linear form on $\mathbb{R}^n$ while f_k may be defined as a linear form on the Cartesian product of two dimensional vector spaces $\mathscr{E}^6$. Having dispelled this ambiguity the distinction in the constraint notation is no longer applied in the rest of the paper.

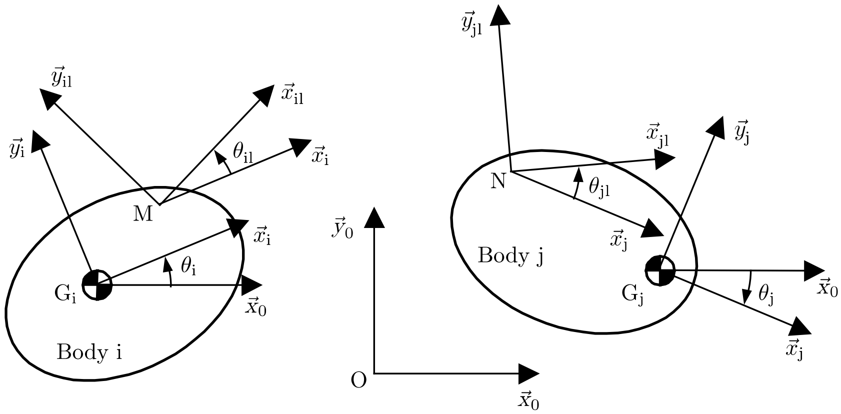

Fig. 4. Geometrical characteristics of two bodies in planar motion.

It is worthwhile noting that all vectors in Eq. (14) must be expressed in the absolute frame of reference.

The constraint terms contributing to the generalized forces are obtained from the general expression given by

$$Q_q = \sum_{k=1}^{m} \frac{\partial f_k}{\partial q} \lambda_k = \sum_{k=1}^{m} \frac{\partial \dot{f}_k}{\partial \dot{q}} \lambda_k \tag{15}$$

Using the generalized coordinates defined for Fig. 4 bodies in planar motion and Eq. (14) for the expression of the kinematic constraint equations, the constraint terms contributing to the generalized forces are

$$\text{for body } i: \begin{cases} Q_{x_i} = \lambda_k \vec{f}_{k,\overrightarrow{OM}} \cdot \vec{x}_0 \\ Q_{y_i} = \lambda_k \vec{f}_{k,\overrightarrow{OM}} \cdot \vec{y}_0 \\ Q_{\theta_i} = -\lambda_k \left(\vec{f}_{k,\vec{x}_{il}} \times \vec{x}_{il} + \vec{f}_{k,\vec{y}_{il}} \times \vec{y}_{il} \right) \cdot \vec{z}_0 + \lambda_k \vec{f}_{k,\overrightarrow{OM}} \cdot \left(\vec{z}_0 \times \overrightarrow{G_i M} \right) \end{cases}$$

$$\text{for body } j: \begin{cases} Q_{x_j} = \lambda_k \vec{f}_{k,\overrightarrow{ON}} \cdot \vec{x}_0 \\ Q_{y_j} = \lambda_k \vec{f}_{k,\overrightarrow{ON}} \cdot \vec{y}_0 \\ Q_{\theta_j} = -\lambda_k \left(\vec{f}_{k,\vec{x}_{jl}} \times \vec{x}_{jl} + \vec{f}_{k,\vec{y}_{jl}} \times \vec{y}_{jl} \right) \cdot \vec{z}_0 + \lambda_k \vec{f}_{k,\overrightarrow{ON}} \cdot \left(\vec{z}_0 \times \overrightarrow{G_j N} \right) \end{cases} \tag{16}$$

with the Einstein implicit summation convention on the repeated subscript k.

By analogy to the case of a given action (Eq. (9)) the following variables can now be clearly expressed:

$$\text{for body } i\text{:}\quad \begin{cases} F_{x_i} = \lambda_k \vec{f}_{k,\overrightarrow{OM}} \cdot \vec{x}_0 \\ F_{y_i} = \lambda_k \vec{f}_{k,\overrightarrow{OM}} \cdot \vec{y}_0 \\ C_{z_i} = -\lambda_k \left(\vec{f}_{k,\vec{x}_{il}} \times \vec{x}_{il} + \vec{f}_{k,\vec{y}_{il}} \times \vec{y}_{il} \right) \cdot \vec{z}_0 \end{cases}$$
$$\text{for body } j\text{:}\quad \begin{cases} F_{x_j} = \lambda_k \vec{f}_{k,\overrightarrow{ON}} \cdot \vec{x}_0 \\ F_{y_j} = \lambda_k \vec{f}_{k,\overrightarrow{ON}} \cdot \vec{y}_0 \\ C_{z_j} = -\lambda_k \left(\vec{f}_{k,\vec{x}_{jl}} \times \vec{x}_{jl} + \vec{f}_{k,\vec{y}_{jl}} \times \vec{y}_{jl} \right) \cdot \vec{z}_0 \end{cases} \tag{17}$$

These variables are calculated in the joint component and passed respectively to body i and body j components that calculate the terms given by relations (18) to complete the generalized forces Q_{θ_i} and Q_{θ_j}.

$$\begin{aligned} &\vec{F}_i \cdot \left(\vec{z}_0 \times \overrightarrow{G_iM} \right) \quad \text{with } \vec{F}_i = F_{x_i}\vec{x}_0 + F_{y_i}\vec{y}_0 \text{ for body } i \\ &\vec{F}_j \cdot \left(\vec{z}_0 \times \overrightarrow{G_jN} \right) \quad \text{with } \vec{F}_j = F_{x_j}\vec{x}_0 + F_{y_j}\vec{y}_0 \text{ for body } j \end{aligned} \tag{18}$$

In turn each body component integrates its dynamic model and furnishes, to the joint component, the absolute position of point M for body i and point N for body j and the absolute angular position of axis $\vec{x}_{il}$ for body i and axis $\vec{x}_{jl}$ for body j (Eqs. (19) and (20)). The absolute position of a point M (respectively, N) is obtained with the absolute position of the body mass center G_i (respectively, G_j) and the relative position vector of this point M (respectively, N) projected onto the absolute frame of reference. The absolute angular position of the axis $\vec{x}_{il}$ (respectively, $\vec{x}_{jl}$) is simply the sum of the body absolute angular position and the relative angular position of this axis $\vec{x}_{il}$ (respectively, $\vec{x}_{jl}$) in the body frame. The relative position vectors of points M and N and the relative angular positions of axes $\vec{x}_{il}$ and $\vec{x}_{jl}$ are parameters in the body components

$$\begin{cases} x_M^0 = x_{G_i} + x_M \cos\theta_i - y_M \sin\theta_i \\ y_M^0 = y_{G_i} + x_M \sin\theta_i + y_M \cos\theta_i \\ \theta_M^0 = \theta_i + \theta_{il} \end{cases} \tag{19}$$

$$\begin{cases} x_N^0 = x_{G_j} + x_N \cos\theta_j - y_N \sin\theta_j \\ y_N^0 = y_{G_j} + x_N \sin\theta_j + y_N \cos\theta_j \\ \theta_N^0 = \theta_j + \theta_{jl} \end{cases} \tag{20}$$

In the joint component the Lagrange multipliers λ_k remain to be calculated. This is addressed later in this section. Before this we will illustrate the above mathematical formulation onto the example of a translational joint.

3.3. Illustration of the joint dynamics formulation on the translational joint example

A translational joint between points M and N of two bodies may be schematically represented by Fig. 5.

The geometrical constraints imposed by a translational joint may be expressed by

$$\begin{aligned} f_1\left(\overrightarrow{OM},\overrightarrow{ON},\vec{y}_{il}\right) &= \overrightarrow{MN}\cdot\vec{y}_{il} = \overrightarrow{ON}\cdot\vec{y}_{il} - \overrightarrow{OM}\cdot\vec{y}_{il} = 0 \\ f_2\left(\overrightarrow{OM},\overrightarrow{ON},\vec{y}_{jl}\right) &= \overrightarrow{MN}\cdot\vec{y}_{jl} = \overrightarrow{ON}\cdot\vec{y}_{jl} - \overrightarrow{OM}\cdot\vec{y}_{jl} = 0 \end{aligned} \qquad (21)$$

An alternative to this formulation is, for instance, $\overrightarrow{MN}\cdot\vec{y}_{il} = 0$ and $\vec{x}_{jl}\cdot\vec{y}_{il} = 0$ but Eqs. (21) enable the symmetry property of the translational joint component to be conserved in its model formulation. Then Eq. (22) give the variables calculated in the translational joint component and passed to the body components (see Eq. (17)).

$$\text{for body } i\text{:} \quad \begin{cases} F_{x_i} = \lambda_1(-\vec{y}_{il})\cdot\vec{x}_0 + \lambda_2(-\vec{y}_{jl})\cdot\vec{x}_0 = \lambda_1\sin\theta_M^0 + \lambda_2\sin\theta_N^0 \\ F_{y_i} = \lambda_1(-\vec{y}_{il})\cdot\vec{y}_0 + \lambda_2(-\vec{y}_{jl})\cdot\vec{y}_0 = -\lambda_1\cos\theta_M^0 - \lambda_2\cos\theta_N^0 \\ C_{z_i} = -\lambda_1\left[\left(\overrightarrow{ON}-\overrightarrow{OM}\right)\times\vec{y}_{il}\right]\cdot\vec{z}_0 \\ \quad\;\; = -\lambda_1\left[\left(x_N^0 - x_M^0\right)\cos\theta_M^0 + \left(y_N^0 - y_M^0\right)\sin\theta_M^0\right] \end{cases}$$

$$\text{for body } j\text{:} \quad \begin{cases} F_{x_j} = \lambda_1\vec{y}_{il}\cdot\vec{x}_0 + \lambda_2\vec{y}_{jl}\cdot\vec{x}_0 = -\lambda_1\sin\theta_M^0 - \lambda_2\sin\theta_N^0 \\ F_{y_j} = \lambda_1\vec{y}_{il}\cdot\vec{y}_0 + \lambda_2\vec{y}_{jl}\cdot\vec{y}_0 = \lambda_1\cos\theta_M^0 + \lambda_2\cos\theta_N^0 \\ C_{z_j} = -\lambda_2\left[\left(\overrightarrow{ON}-\overrightarrow{OM}\right)\times\vec{y}_{jl}\right]\cdot\vec{z}_0 \\ \quad\;\; = -\lambda_2\left[\left(x_N^0 - x_M^0\right)\cos\theta_N^0 + \left(y_N^0 - y_M^0\right)\sin\theta_N^0\right] \end{cases} \qquad (22)$$

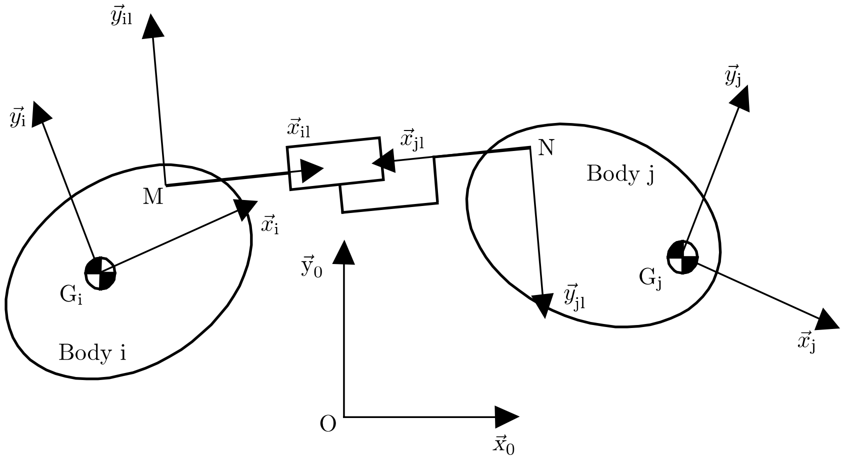

Fig. 5. Schematic representation of a translational joint between two bodies.

Furthermore the generalized forces due to the constraint terms in the body component mathematical models are calculated with Eqs. (16).

Until this point bodies with only one connecting point have been considered. Formulation development is strictly identical, as in those seen above, for bodies with several connecting points. Previous calculation is only lengthened proportionally to the number of connecting points. The generalized forces due to different constraints in this case are simply summed up for the number of connecting ports.

Now the calculation of the Lagrange multipliers is addressed.

3.4. Calculation of the Lagrange multipliers

This formulation, in dependent coordinates, led to the use of the Lagrange multiplier method. The Lagrange multipliers are implicit solutions of the constraint equations embedded in the joint components and thus must be calculated in these components. AMESim enables implicit equations to be programmed, the dynamics equations, however, globally obtained on a complete system, are generally a Differential–Algebraic Equation system of index 3 [10,22] with the geometrical constraints programmed in this way. This Differential–Algebraic Equation index causes serious and even critical problems for numerical resolution. Implementation of kinematic constraint equations or even the second derivative with respect to time (acceleration order) of the geometrical constraint equations would decrease the index. However this also causes a loss of information at the geometrical or even kinematic level and the numerical solution has a high chance of diverging from the exact solution.

In order to circumvent this problem, the Baumgarte stabilization [3] is used to calculate the Lagrange multipliers in an implicit manner. The Baumgarte stabilization is based on a control scheme [20] of the constraint errors to make them converge towards zero. Its calculus scheme is a PID control on the kinematic constraint equations which then also use the constraint equations at the geometrical and acceleration orders (e.g. [8]). In this manner all the information about the constraints is kept and the judicious choice of the PID schema gains enables the constraint errors to converge towards zero.

If Eq. (13) represents the geometrical constraint equations then the Baumgarte stabilization calculus scheme is given by:

$$\ddot{f}_k + 2\alpha\dot{f}_k + \beta^2 f_k = 0 \quad \text{for } k = 1 \text{ to } m \tag{23}$$

where α/β and β may, respectively, be interpreted as a damping coefficient and a natural pulsation of the solution for f_k. The same numerical coefficients have been arbitrarily chosen for all the constraint equations.

The use of this stabilization scheme obliges the body component to supply another piece of information other than the information concerning the point M position and on the angular position of the joint axis. In fact the kinematic and the acceleration constraint equations are obtained from the differentiation, with respect to time, in the absolute frame of reference of the geometrical constraint equation (13). The first differentiation gives (see the appendix for calculation details and notations)

$$\vec{f}_{k,\overrightarrow{OM}} \cdot \left(\dot{x}^0_M \vec{x}_0 + \dot{y}^0_M \vec{y}_0\right) + \vec{f}_{k,\overrightarrow{ON}} \cdot \left(\dot{x}^0_N \vec{x}_0 + \dot{y}^0_N \vec{y}_0\right) + \dot{\theta}_i \left(\vec{f}_{k,\vec{x}_{il}} \cdot \vec{y}_{il} - \vec{f}_{k,\vec{y}_{il}} \cdot \vec{x}_{il}\right)$$
$$+ \dot{\theta}_j \left(\vec{f}_{k,\vec{x}_{jl}} \cdot \vec{y}_{jl} - \vec{f}_{k,\vec{y}_{jl}} \cdot \vec{x}_{jl}\right) = 0 \quad \text{for } k = 1 \text{ to } m \tag{24}$$

Eq. (24) shows that body components must also supply the absolute velocities of points M and point N and the absolute angular velocities of both bodies (which are equal to the absolute angular velocities of the joint axes). These quantities for point M (respectively, for point N by substituting i by j and M by N) are given by

$$\begin{cases} \dot{x}^0_M = \dot{x}_{G_i} - \dot{\theta}_i (x_M \sin\theta_i + y_M \cos\theta_i) \\ \dot{y}^0_M = \dot{y}_{G_i} + \dot{\theta}_i (x_M \cos\theta_i - y_M \sin\theta_i) \end{cases} \tag{25}$$

The second differentiation, with respect to time, in the absolute frame of reference of the geometrical constraint equations gives (see the appendix for calculation details and notations):

$$\vec{f}_{k,\overrightarrow{OM}} \cdot \left(\ddot{x}^0_M \vec{x}_0 + \ddot{y}^0_M \vec{y}_0\right) + \vec{f}_{k,\overrightarrow{ON}} \cdot \left(\ddot{x}^0_N \vec{x}_0 + \ddot{y}^0_N \vec{y}_0\right) + f_{k,\overrightarrow{OM}^2} \left(\dot{x}^{0\,2}_M + \dot{y}^{0\,2}_M\right)$$
$$+ f_{k,\overrightarrow{ON}^2} \left(\dot{x}^{0\,2}_N + \dot{y}^{0\,2}_N\right) + 2 f_{k,\overrightarrow{OM},\overrightarrow{ON}} \left(\dot{x}^0_M \dot{x}^0_N + \dot{y}^0_M \dot{y}^0_N\right) + \ddot{\theta}_i \left(\vec{f}_{k,\vec{x}_{il}} \cdot \vec{y}_{il} - \vec{f}_{k,\vec{y}_{il}} \cdot \vec{x}_{il}\right)$$
$$+ \ddot{\theta}_j \left(\vec{f}_{k,\vec{x}_{jl}} \cdot \vec{y}_{jl} - \vec{f}_{k,\vec{y}_{jl}} \cdot \vec{x}_{jl}\right) - \dot{\theta}^2_i \left(\vec{f}_{k,\vec{x}_{il}} \cdot \vec{x}_{il} + \vec{f}_{k,\vec{y}_{il}} \cdot \vec{y}_{il}\right) - \dot{\theta}^2_j \left(\vec{f}_{k,\vec{x}_{jl}} \cdot \vec{x}_{jl} + \vec{f}_{k,\vec{y}_{jl}} \cdot \vec{y}_{jl}\right)$$
$$+ 2\dot{\theta}_i \dot{\theta}_j \left[\left(f_{k,\vec{x}_{il},\vec{x}_{jl}} + f_{k,\vec{y}_{il},\vec{y}_{jl}}\right) \cos\left(\theta^0_M - \theta^0_N\right) - \left(f_{k,\vec{x}_{il},\vec{y}_{jl}} - f_{k,\vec{y}_{il},\vec{x}_{jl}}\right) \sin\left(\theta^0_M - \theta^0_N\right)\right]$$
$$- 2\dot{x}^0_M \left[\dot{\theta}_i \left(f_{k,\overrightarrow{OM},\vec{x}_{il}} \sin\theta^0_M + f_{k,\overrightarrow{OM},\vec{y}_{il}} \cos\theta^0_M\right) + \dot{\theta}_j \left(f_{k,\overrightarrow{OM},\vec{x}_{jl}} \sin\theta^0_N + f_{k,\overrightarrow{OM},\vec{y}_{jl}} \cos\theta^0_N\right)\right]$$
$$+ 2\dot{y}^0_M \left[\dot{\theta}_i \left(f_{k,\overrightarrow{OM},\vec{x}_{il}} \cos\theta^0_M - f_{k,\overrightarrow{OM},\vec{y}_{il}} \sin\theta^0_M\right) + \dot{\theta}_j \left(f_{k,\overrightarrow{OM},\vec{x}_{jl}} \cos\theta^0_N - f_{k,\overrightarrow{OM},\vec{y}_{jl}} \sin\theta^0_N\right)\right]$$
$$- 2\dot{x}^0_N \left[\dot{\theta}_i \left(f_{k,\overrightarrow{ON},\vec{x}_{il}} \sin\theta^0_M + f_{k,\overrightarrow{ON},\vec{y}_{il}} \cos\theta^0_M\right) + \dot{\theta}_j \left(f_{k,\overrightarrow{ON},\vec{x}_{jl}} \sin\theta^0_N + f_{k,\overrightarrow{ON},\vec{y}_{jl}} \cos\theta^0_N\right)\right]$$
$$+ 2\dot{y}^0_N \left[\dot{\theta}_i \left(f_{k,\overrightarrow{ON},\vec{x}_{il}} \cos\theta^0_M - f_{k,\overrightarrow{ON},\vec{y}_{il}} \sin\theta^0_M\right) + \dot{\theta}_j \left(f_{k,\overrightarrow{ON},\vec{x}_{jl}} \cos\theta^0_N - f_{k,\overrightarrow{ON},\vec{y}_{jl}} \sin\theta^0_N\right)\right]$$
$$= 0 \quad \text{for } k = 1 \text{ to } m \tag{26}$$

Eq. (26) shows in turn that the absolute accelerations of points M and N and the absolute angular accelerations of both bodies (equal to the absolute angular accelerations of the joint axes) are needed for a joint component model where Baumgarte stabilization is implemented. The acceleration of point M (respectively, for point N by substituting i by j and M by N) is given by

$$\begin{cases} \ddot{x}^0_M = \ddot{x}_{G_i} - \ddot{\theta}_i (x_M \sin\theta_i + y_M \cos\theta_i) - \dot{\theta}^2_i (x_M \cos\theta_i - y_M \sin\theta_i) \\ \ddot{y}^0_M = \ddot{y}_{G_i} + \ddot{\theta}_i (x_M \cos\theta_i - y_M \sin\theta_i) - \dot{\theta}^2_i (x_M \sin\theta_i + y_M \cos\theta_i) \end{cases} \tag{27}$$

The peculiarity of each joint generally eliminates a number of terms in these equations. Nevertheless Eqs. (24) and (26) enable generic kinematic and acceleration constraint equations to be expressed in terms of the variables supplied to a joint component.

The kinematic and acceleration constraint equations are now illustrated on the translational joint example. The geometrical constraint equations in this case are given by Eqs. (21). In terms of the variables passed at the connecting port of the joint component these equations become:

$$\begin{cases} -\left(x_N^0 - x_M^0\right)\sin\theta_M^0 + \left(y_N^0 - y_M^0\right)\cos\theta_M^0 = 0 \\ -\left(x_N^0 - x_M^0\right)\sin\theta_N^0 + \left(y_N^0 - y_M^0\right)\cos\theta_N^0 = 0 \end{cases} \tag{28}$$

The corresponding kinematic constraint equations are

$$\begin{cases} -\vec{y}_{il}\cdot\vec{V}^0(M) + \vec{y}_{il}\cdot\vec{V}^0(N) + \left(-\overrightarrow{OM} + \overrightarrow{ON}\right)\cdot\left(\vec{\Omega}_i^0 \times \vec{y}_{il}\right) = 0 \\ -\vec{y}_{jl}\cdot\vec{V}^0(M) + \vec{y}_{jl}\cdot\vec{V}^0(N) + \left(-\overrightarrow{OM} + \overrightarrow{ON}\right)\cdot\left(\vec{\Omega}_j^0 \times \vec{y}_{jl}\right) = 0 \end{cases}$$

$$\Longleftrightarrow \begin{cases} -\left(\dot{x}_N^0 - \dot{x}_M^0\right)\sin\theta_M^0 + \left(\dot{y}_N^0 - \dot{y}_M^0\right)\cos\theta_M^0 - \dot{\theta}_i\left[\left(x_N^0 - x_M^0\right)\cos\theta_M^0\right. \\ \quad \left. + \left(y_N^0 - y_M^0\right)\sin\theta_M^0\right] = 0 \\ -\left(\dot{x}_N^0 - \dot{x}_M^0\right)\sin\theta_N^0 + \left(\dot{y}_N^0 - \dot{y}_M^0\right)\cos\theta_N^0 - \dot{\theta}_j\left[\left(x_N^0 - x_M^0\right)\cos\theta_N^0\right. \\ \quad \left. + \left(y_N^0 - y_M^0\right)\sin\theta_N^0\right] = 0 \end{cases} \tag{29}$$

Finally the acceleration constraint equations are

$$\begin{cases} -\vec{y}_{il}\cdot\vec{J}^0(M) + \vec{y}_{il}\cdot\vec{J}^0(N) + \vec{\gamma}_i^0\cdot\left(\vec{y}_{il}\times\overrightarrow{MN}\right) + 2\left[\vec{\Omega}_i^0 \times (-\vec{y}_{il})\right]\cdot\vec{V}^0(M) \\ \quad + 2\left[\vec{\Omega}_i^0 \times \vec{y}_{il}\right]\cdot\vec{V}^0(N) = 0 \\ -\vec{y}_{jl}\cdot\vec{J}^0(M) + \vec{y}_{jl}\cdot\vec{J}^0(N) + \vec{\gamma}_j^0\cdot\left(\vec{y}_{jl}\times\overrightarrow{MN}\right) + 2\left[\vec{\Omega}_j^0 \times (-\vec{y}_{jl})\right]\cdot\vec{V}^0(M) \\ \quad + 2\left[\vec{\Omega}_j^0 \times \vec{y}_{jl}\right]\cdot\vec{V}^0(N) = 0 \end{cases}$$

$$\Longleftrightarrow \begin{cases} -\left(\ddot{x}_N^0 - \ddot{x}_M^0\right)\sin\theta_M^0 + \left(\ddot{y}_N^0 - \ddot{y}_M^0\right)\cos\theta_M^0 - \ddot{\theta}_i\left[\left(x_N^0 - x_M^0\right)\cos\theta_M^0\right. \\ \quad \left. + \left(y_N^0 - y_M^0\right)\sin\theta_M^0\right] + 2\dot{\theta}_i\left[-\left(\dot{x}_N^0 - \dot{x}_M^0\right)\cos\theta_M^0 - \left(\dot{y}_N^0 - \dot{y}_M^0\right)\sin\theta_M^0\right] = 0 \\ -\left(\ddot{x}_N^0 - \ddot{x}_M^0\right)\sin\theta_N^0 + \left(\ddot{y}_N^0 - \ddot{y}_M^0\right)\cos\theta_N^0 - \ddot{\theta}_j\left[\left(x_N^0 - x_M^0\right)\cos\theta_N^0\right. \\ \quad \left. + \left(y_N^0 - y_M^0\right)\sin\theta_N^0\right] + 2\dot{\theta}_j\left[-\left(\dot{x}_N^0 - \dot{x}_M^0\right)\cos\theta_N^0 - \left(\dot{y}_N^0 - \dot{y}_M^0\right)\sin\theta_N^0\right] = 0 \end{cases} \tag{30}$$

It is worthwhile noting in Eq. (30) that the squared terms of the absolute angular velocities of bodies vanish by using the geometrical constraint equations. Some terms will often vanish, particularly for the joints presented in the library.

4. Conclusion

The purpose of this paper is not to develop a new mechanical theory or a new formulation for writing dynamics equations of planar multibody mechanical systems (one may find valuable information on this topic in [19] or [12]). Instead the issue was more to fit existing formulations to the AMESim requirements in order to build a new library in the domain of planar mechanical systems. The tool requirements were principally modularity and communicability between components which must have predefined mathematical models, in particular between body and joint components. This has led to a formulation in dependant coordinates, so the Lagrange multiplier method was used. The generalized coordinates are the absolute Cartesian coordinates of the body mass centers and the absolute angular positions of the bodies. The body component mathematical model consists of the Lagrange equations written in terms of the previously given generalized coordinates. The joint components consist of joint constraint equations from which the Lagrange multipliers, as implicit solutions, are numerically obtained. Baumgarte stabilization schemas have been implemented onto the constraint equations in order to decrease the Differential–Algebraic Equation index without degrading the numerical solution.

The main originality of the mathematical model implementation, caused by Amesim requirements, consists of geometrical constraint equations explicitly expressed in terms of vectors and exploited in this form (Eq. (13)) instead of expressed in terms of scalars (Eq. (10)). This was required to show the generic formulation of constraint actions and the variables received by body components (Eq. (9)), whatever the connected joint component. In fact Eq. (10) would not have enabled terms (18) to be developed, these terms are to be treated in the body component.

The generic formulation proposed in this paper (in particular for joint constraints), though dedicated to causal component mathematical models, may be used for systematically deriving the equations associated with joint constraints. One can even imagine a symbolical routine that generates these equations with the vector constraint as the only input. Concerning the use of Baumgarte's stabilization with Lagrange's equations and the Lagrange multiplier method, the formulation is more classical. Here it shows how to split the different mathematical submodels according to the modular and component approach of an object-oriented tool. Also it is quite well adapted to tools that do not globally re-oriente in terms of variable assignment and re-organize the mathematical model. Moreover, in the context of object-oriented tools, one advantage of the formulation presented is that the generated Differential–Algebraic Equation system is generally of index one even for closed loop structures. In fact, by using Baumgarte's stabilization, it was not needed to reformulate a posteriori the model that otherwise would have been at least of index three. The drawback to this approach is that the size of the state model is increased and it has, in addition, certain number of algebraic equations and implicit variables.

One logical perspective to the formulation proposed is its extrapolation to three dimensional mechanisms. There is no conceptual difficulty for using the same scheme by combining Lagrange's equations, the Lagrange multiplier method and Baumgarte's stabilization as well as splitting the models into the same components, namely bodies and joints. Basically the main difficulty is intrinsic to the three dimensional mechanism domain. This renders the generic formulation proposed in this paper more tedious. The main reason resides in the necessity of taking the body orientation into account. For planar mechanisms only the rotation around one absolute reference fixed axis was needed. This has greatly simplified some mathematical developments in the formulation. With three dimensional mechanisms it is necessary to choose a set of generalized coordinates to represent the absolute orientations of bodies. Euler's angles or Euler's parameters are examples of such generalized coordinates. With three of Euler's angles the main drawback is the possible occurrence of singularities where there is either non unique solutions or discontinuities. It is then necessary to switch to another set of Euler's angles. Otherwise Euler's parameters are well designed for these cases but there are four of them and so this requires an extra algebraic relation to be taken into account in the formulation [24].

Part II of this paper details the library composition and illustrates it with the example of a seven-body mechanism [23]. It also shows how actuating items are included in the joint components.

Appendix A. Nomenclature

$\vec{a} \cdot \vec{b}$	scalar product between two vectors
$\vec{a} \times \vec{b}$	cross product between two vectors
$\dot{a} = \frac{\mathrm{d}a}{\mathrm{d}t},\ \ddot{a} = \frac{\mathrm{d}^2 a}{\mathrm{d}t^2}$	first and second time differentiations
$\frac{\partial a}{\partial b}$	partial derivative
a^*	virtual quantities
$\vec{a}$	vector
$\underline{a}$	column vector
a^{T}	matrix transpose
A^*	virtual power developed by acceleration quantities
A	acceleration quantities
T	kinetic coenergy
P^*	virtual power developed by mechanical actions
Q	generalized forces
$(\mathrm{O}, \vec{x}_0, \vec{y}_0, \vec{z}_0)$	absolute frame of reference
$(G_i, \vec{x}_i, \vec{y}_i, \vec{z}_i)$	body frame
$(M_i, \vec{x}_{il}, \vec{y}_{il}, \vec{z}_{il})$	joint privileged frame
(x_{G_i}, y_{G_i})	absolute coordinates of the body mass center
θ_i	absolute angular body position
(x_{M_i}, y_{M_i})	relative coordinates of point M_i
θ_{il}	relative angular position of joint privileged frame
$(x^0_{M_i}, y^0_{M_i})$	absolute coordinates of point M_i
θ^0_M	absolute angular position of joint privileged frame
m_i, I_i	body mass, central moment of inertia about $\vec{z}_0$
g	gravity acceleration
$\{\mathscr{W}\}$	wrench
$\vec{F}$	force
(F_x, F_y)	force absolute coordinates
$\vec{M}(M)$	torque about point M

C_z	torque component on $\vec{z}_0$
$\vec{V}^0(M)$	absolute velocity of point M
$\vec{J}^0(M)$	absolute acceleration of point M
$\vec{\Omega}_i^0$	absolute angular velocity of body i
$\vec{\gamma}_i^0$	absolute angular acceleration of body i
q	generalized coordinates
g_k, f_k	geometrical constraint expressions
λ_k	Lagrange multipliers
α, β	parameters of the Baumgarte stabilization

Appendix B. Notation conventions on partial derivatives with respect to a vector

We introduce here a conventional notation for sake of conciseness. The vector noted $\vec{f}_{\vec{a}} = \frac{\partial f}{\partial \vec{a}}$ represents the partial derivative in the absolute frame of reference of f with respect to the vector $\vec{a}$. We suppose here that the constraint equations only consist of sums of scalar products of the form $\alpha\vec{a} \cdot \vec{b}$. In this case $\vec{f}_{\vec{a}} = \alpha\vec{b}$. The vector components of $\vec{f}_{\vec{a}}$ are the partial derivatives of f with respect to the vector components of $\vec{a}$. With the scalar product properties, we may manipulate the partial derivatives with respect to a vector as we do with partial derivatives with respect to scalars. Furthermore the partial derivative of $\vec{f}_{\vec{a}}$ with respect to a second vector is a scalar noted $f_{\vec{a},\vec{b}} = \frac{\partial \vec{f}_{\vec{a}}}{\partial \vec{b}}$. In the same context, for constraint equations consisting only of sums of scalar products $f_{\vec{a},\vec{b}} = f_{\vec{b},\vec{a}} = \alpha$. The notation introduced here differs from that in [19] for instance, in the sense that $\vec{f}_{\vec{a}}$ is not treated as a row vector but as a full vector. Finally we denote $f_{\vec{a}^2} = f_{\vec{a},\vec{a}}$.

Appendix C. Development of the kinematic and acceleration constraint equations

The kinematic constraint equations are obtained by differentiation with respect to time in the absolute frame of reference of Eq. (13) (see appendix on notation conventions):

$$\frac{\mathrm{d}f_k\left(\overrightarrow{OM},\overrightarrow{ON},\vec{x}_{il},\vec{y}_{il},\vec{x}_{jl},\vec{y}_{jl}\right)}{\mathrm{d}t} = 0 \quad \text{for } k = 1 \text{ to } m$$

$$\Longleftrightarrow \frac{\partial f_k}{\partial \overrightarrow{OM}} \cdot \frac{\mathrm{d}\overrightarrow{OM}}{\mathrm{d}t} + \frac{\partial f_k}{\partial \overrightarrow{ON}} \cdot \frac{\mathrm{d}\overrightarrow{ON}}{\mathrm{d}t} + \frac{\partial f_k}{\partial \vec{x}_{il}} \cdot \frac{\mathrm{d}\vec{x}_{il}}{\mathrm{d}t} + \frac{\partial f_k}{\partial \vec{y}_{il}} \cdot \frac{\mathrm{d}\vec{y}_{il}}{\mathrm{d}t} + \frac{\partial f_k}{\partial \vec{x}_{jl}} \cdot \frac{\mathrm{d}\vec{x}_{jl}}{\mathrm{d}t} + \frac{\partial f_k}{\partial \vec{y}_{jl}} \cdot \frac{\mathrm{d}\vec{y}_{jl}}{\mathrm{d}t} = 0 \quad \text{for } k = 1 \text{ to } m$$

$$\Longleftrightarrow \vec{f}_{k,\overrightarrow{OM}} \cdot \vec{V}^0(M) + \vec{f}_{k,\overrightarrow{ON}} \cdot \vec{V}^0(N) + \vec{f}_{k,\vec{x}_{il}} \cdot \left(\vec{\Omega}_i^0 \times \vec{x}_{il}\right) + \vec{f}_{k,\vec{y}_{il}} \cdot \left(\vec{\Omega}_i^0 \times \vec{y}_{il}\right) + \vec{f}_{k,\vec{x}_{jl}} \cdot \left(\vec{\Omega}_j^0 \times \vec{x}_{jl}\right) + \vec{f}_{k,\vec{y}_{jl}} \cdot \left(\vec{\Omega}_j^0 \times \vec{y}_{jl}\right) = 0 \quad \text{for } k = 1 \text{ to } m \tag{31}$$

These equations are used for the joint component generic model (Eq. (24)). The constraint terms used in the body component are obtained by expressing Eq. (31) in terms of the generalized velocities. Using the velocity transport from points M (respectively, N) to points G_i (respectively, G_j) achieves this.

$$\begin{aligned}
&\vec{f}_{k,\overrightarrow{OM}} \cdot \left(\vec{V}^0(G_i) + \vec{\Omega}_i^0 \times \overrightarrow{G_iM}\right) + \vec{f}_{k,\overrightarrow{ON}} \cdot \left(\vec{V}^0(G_j) + \vec{\Omega}_j^0 \times \overrightarrow{G_jN}\right) + \vec{f}_{k,\vec{x}_{il}} \\
&\cdot \left(\vec{\Omega}_i^0 \times \vec{x}_{il}\right) + \vec{f}_{k,\vec{y}_{il}} \cdot \left(\vec{\Omega}_i^0 \times \vec{y}_{il}\right) + \vec{f}_{k,\vec{x}_{jl}} \cdot \left(\vec{\Omega}_j^0 \times \vec{x}_{jl}\right) \\
&+ \vec{f}_{k,\vec{y}_{jl}} \cdot \left(\vec{\Omega}_j^0 \times \vec{y}_{jl}\right) = 0 \quad \text{for } k = 1 \text{ to } m \\
&\Longleftrightarrow \vec{f}_{k,\overrightarrow{OM}} \cdot \vec{V}^0(G_i) + \vec{f}_{k,\overrightarrow{ON}} \cdot \vec{V}^0(G_j) \\
&\quad + \left(\overrightarrow{G_iM} \times \vec{f}_{k,\overrightarrow{OM}} - \vec{f}_{k,\vec{x}_{il}} \times \vec{x}_{il} - \vec{f}_{k,\vec{y}_{il}} \times \vec{y}_{il}\right) \cdot \vec{\Omega}_i^0 \\
&\quad + \left(\overrightarrow{G_jN} \times \vec{f}_{k,\overrightarrow{ON}} - \vec{f}_{k,\vec{x}_{jl}} \times \vec{x}_{jl} - \vec{f}_{k,\vec{y}_{jl}} \times \vec{y}_{jl}\right) \cdot \vec{\Omega}_j^0 = 0 \quad \text{for } k = 1 \text{ to } m
\end{aligned}$$

which leads to Eq. (14).

Starting again from Eq. (31), their differentiation with respect to time in the absolute frame of reference gives

$$\begin{aligned}
&\vec{f}_{k,\overrightarrow{OM}} \cdot \vec{J}^0(M) + \vec{f}_{k,\overrightarrow{ON}} \cdot \vec{J}^0(N) + \frac{\partial \vec{f}_{k,\overrightarrow{OM}}}{\partial \overrightarrow{OM}} \frac{\mathrm{d}\overrightarrow{OM}}{\mathrm{d}t} \cdot \vec{V}^0(M) + \frac{\partial \vec{f}_{k,\overrightarrow{ON}}}{\partial \overrightarrow{ON}} \frac{\mathrm{d}\overrightarrow{ON}}{\mathrm{d}t} \cdot \vec{V}^0(N) \\
&+ \frac{\partial \vec{f}_{k,\overrightarrow{OM}}}{\partial \overrightarrow{ON}} \frac{\mathrm{d}\overrightarrow{ON}}{\mathrm{d}t} \cdot \vec{V}^0(M) + \frac{\partial \vec{f}_{k,\overrightarrow{ON}}}{\partial \overrightarrow{OM}} \frac{\mathrm{d}\overrightarrow{OM}}{\mathrm{d}t} \cdot \vec{V}^0(N) \\
&+ \left(\frac{\partial \vec{f}_{k,\overrightarrow{OM}}}{\partial \vec{x}_{il}} \frac{\mathrm{d}\vec{x}_{il}}{\mathrm{d}t} + \frac{\partial \vec{f}_{k,\overrightarrow{OM}}}{\partial \vec{y}_{il}} \frac{\mathrm{d}\vec{y}_{il}}{\mathrm{d}t} + \frac{\partial \vec{f}_{k,\overrightarrow{OM}}}{\partial \vec{x}_{jl}} \frac{\mathrm{d}\vec{x}_{jl}}{\mathrm{d}t} + \frac{\partial \vec{f}_{k,\overrightarrow{OM}}}{\partial \vec{y}_{jl}} \frac{\mathrm{d}\vec{y}_{jl}}{\mathrm{d}t}\right) \cdot \vec{V}^0(M) \\
&+ \left(\frac{\partial \vec{f}_{k,\overrightarrow{ON}}}{\partial \vec{x}_{il}} \frac{\mathrm{d}\vec{x}_{il}}{\mathrm{d}t} + \frac{\partial \vec{f}_{k,\overrightarrow{ON}}}{\partial \vec{y}_{il}} \frac{\mathrm{d}\vec{y}_{il}}{\mathrm{d}t} + \frac{\partial \vec{f}_{k,\overrightarrow{ON}}}{\partial \vec{x}_{jl}} \frac{\mathrm{d}\vec{x}_{jl}}{\mathrm{d}t} + \frac{\partial \vec{f}_{k,\overrightarrow{ON}}}{\partial \vec{y}_{jl}} \frac{\mathrm{d}\vec{y}_{jl}}{\mathrm{d}t}\right) \cdot \vec{V}^0(N) \\
&+ \vec{f}_{k,\vec{x}_{il}} \cdot \frac{\mathrm{d}\left(\vec{\Omega}_i^0 \times \vec{x}_{il}\right)}{\mathrm{d}t} + \vec{f}_{k,\vec{y}_{il}} \cdot \frac{\mathrm{d}\left(\vec{\Omega}_i^0 \times \vec{y}_{il}\right)}{\mathrm{d}t} + \vec{f}_{k,\vec{x}_{jl}} \cdot \frac{\mathrm{d}\left(\vec{\Omega}_j^0 \times \vec{x}_{jl}\right)}{\mathrm{d}t} + \vec{f}_{k,\vec{y}_{jl}} \cdot \frac{\mathrm{d}\left(\vec{\Omega}_j^0 \times \vec{y}_{jl}\right)}{\mathrm{d}t} \\
&+ \left(\frac{\partial \vec{f}_{k,\vec{x}_{il}}}{\partial \overrightarrow{OM}} \frac{\mathrm{d}\overrightarrow{OM}}{\mathrm{d}t} + \frac{\partial \vec{f}_{k,\vec{x}_{il}}}{\partial \overrightarrow{ON}} \frac{\mathrm{d}\overrightarrow{ON}}{\mathrm{d}t} + \frac{\partial \vec{f}_{k,\vec{x}_{il}}}{\partial \vec{x}_{jl}} \frac{\mathrm{d}\vec{x}_{jl}}{\mathrm{d}t} + \frac{\partial \vec{f}_{k,\vec{x}_{il}}}{\partial \vec{y}_{jl}} \frac{\mathrm{d}\vec{y}_{jl}}{\mathrm{d}t}\right) \cdot \left(\vec{\Omega}_i^0 \times \vec{x}_{il}\right) \\
&+ \left(\frac{\partial \vec{f}_{k,\vec{y}_{il}}}{\partial \overrightarrow{OM}} \frac{\mathrm{d}\overrightarrow{OM}}{\mathrm{d}t} + \frac{\partial \vec{f}_{k,\vec{y}_{il}}}{\partial \overrightarrow{ON}} \frac{\mathrm{d}\overrightarrow{ON}}{\mathrm{d}t} + \frac{\partial \vec{f}_{k,\vec{y}_{il}}}{\partial \vec{x}_{jl}} \frac{\mathrm{d}\vec{x}_{jl}}{\mathrm{d}t} + \frac{\partial \vec{f}_{k,\vec{y}_{il}}}{\partial \vec{y}_{jl}} \frac{\mathrm{d}\vec{y}_{jl}}{\mathrm{d}t}\right) \cdot \left(\vec{\Omega}_i^0 \times \vec{y}_{il}\right) \\
&+ \left(\frac{\partial \vec{f}_{k,\vec{x}_{jl}}}{\partial \overrightarrow{OM}} \frac{\mathrm{d}\overrightarrow{OM}}{\mathrm{d}t} + \frac{\partial \vec{f}_{k,\vec{x}_{jl}}}{\partial \overrightarrow{ON}} \frac{\mathrm{d}\overrightarrow{ON}}{\mathrm{d}t} + \frac{\partial \vec{f}_{k,\vec{x}_{jl}}}{\partial \vec{x}_{il}} \frac{\mathrm{d}\vec{x}_{il}}{\mathrm{d}t} + \frac{\partial \vec{f}_{k,\vec{x}_{jl}}}{\partial \vec{y}_{il}} \frac{\mathrm{d}\vec{y}_{il}}{\mathrm{d}t}\right) \cdot \left(\vec{\Omega}_j^0 \times \vec{x}_{jl}\right)
\end{aligned}$$

$$+\left(\frac{\partial\vec{f}_{k,\vec{y}_{jl}}}{\partial\overrightarrow{OM}}\frac{\mathrm{d}\overrightarrow{OM}}{\mathrm{d}t}+\frac{\partial\vec{f}_{k,\vec{y}_{jl}}}{\partial\overrightarrow{ON}}\frac{\mathrm{d}\overrightarrow{ON}}{\mathrm{d}t}+\frac{\partial\vec{f}_{k,\vec{y}_{jl}}}{\partial\vec{x}_{il}}\frac{\mathrm{d}\vec{x}_{il}}{\mathrm{d}t}+\frac{\partial\vec{f}_{k,\vec{y}_{jl}}}{\partial\vec{y}_{il}}\frac{\mathrm{d}\vec{y}_{il}}{\mathrm{d}t}\right)\cdot\left(\vec{\Omega}_j^0\times\vec{y}_{jl}\right)$$
$$=0\quad\text{for } k=1 \text{ to } m$$

Using the hypothesis of constraint equations only constituted by a sum of vector scalar products, we may also state that these scalar products never imply axes of the same joint privileged frame (e.g. $\vec{x}_{il}\cdot\vec{x}_{il}$ or $\vec{x}_{il}\cdot\vec{y}_{il}$ and the same with substitution of *i* by *j*). This explains that terms such as $\frac{\partial\vec{f}_{k,\vec{x}_{il}}}{\partial\vec{x}_{il}}$, $\frac{\partial\vec{f}_{k,\vec{x}_{il}}}{\partial\vec{y}_{il}}$, $\frac{\partial\vec{f}_{k,\vec{y}_{il}}}{\partial\vec{x}_{il}}$ and $\frac{\partial\vec{f}_{k,\vec{y}_{il}}}{\partial\vec{y}_{il}}$ (and the same with substitution of *i* by *j*) do not appear in the previous equations. Using again the notation convention of the partial derivatives with respect to a vector we obtain

$$\vec{f}_{k,\overrightarrow{OM}}\cdot\vec{J}^0(M)+\vec{f}_{k,\overrightarrow{OM}}\cdot\vec{J}^0(N)+f_{k,\overrightarrow{OM}^2}\left[\vec{V}^0(M)\right]^2+f_{k,\overrightarrow{OM}^2}\left[\vec{V}^0(N)\right]^2$$
$$+2f_{k,\overrightarrow{OM},\overrightarrow{ON}}\vec{V}^0(M)\cdot\vec{V}^0(N)+2\left[f_{k,\overrightarrow{OM},\vec{x}_{il}}\left(\vec{\Omega}_i^0\times\vec{x}_{il}\right)\right.$$
$$+f_{k,\overrightarrow{OM},\vec{y}_{il}}\left(\vec{\Omega}_i^0\times\vec{y}_{il}\right)+f_{k,\overrightarrow{OM},\vec{x}_{jl}}\left(\vec{\Omega}_j^0\times\vec{x}_{jl}\right)$$
$$\left.+f_{k,\overrightarrow{OM},\vec{y}_{jl}}\left(\vec{\Omega}_j^0\times\vec{y}_{jl}\right)\right]\cdot\vec{V}^0(M)+2\left[f_{k,\overrightarrow{OM},\vec{x}_{il}}\left(\vec{\Omega}_i^0\times\vec{x}_{il}\right)\right.$$
$$+f_{k,\overrightarrow{ON},\vec{y}_{il}}\left(\vec{\Omega}_i^0\times\vec{y}_{il}\right)+f_{k,\overrightarrow{ON},\vec{x}_{jl}}\left(\vec{\Omega}_j^0\times\vec{x}_{jl}\right)$$
$$\left.+f_{k,\overrightarrow{ON},\vec{y}_{jl}}\left(\vec{\Omega}_j^0\times\vec{y}_{jl}\right)\right]\cdot\vec{V}^0(N)+\vec{f}_{k,\vec{x}_{il}}\left[\left(\vec{\gamma}_i^0\times\vec{x}_{il}\right)+\vec{\Omega}_i^0\times\left(\vec{\Omega}_i^0\times\vec{x}_{il}\right)\right]$$
$$+\vec{f}_{k,\vec{y}_{il}}\left[\left(\vec{\gamma}_i^0\times\vec{y}_{il}\right)+\vec{\Omega}_i^0\times\left(\vec{\Omega}_i^0\times\vec{y}_{il}\right)\right]+\vec{f}_{k,\vec{x}_{jl}}\left[\left(\vec{\gamma}_j^0\times\vec{x}_{jl}\right)+\vec{\Omega}_j^0\times\left(\vec{\Omega}_j^0\times\vec{x}_{jl}\right)\right]$$
$$+\vec{f}_{k,\vec{y}_{jl}}\left[\left(\vec{\gamma}_j^0\times\vec{y}_{jl}\right)+\vec{\Omega}_j^0\times\left(\vec{\Omega}_j^0\times\vec{y}_{jl}\right)\right]$$
$$+\left[f_{k,\vec{x}_{il},\vec{x}_{jl}}\left(\vec{\Omega}_j^0\times\vec{x}_{jl}\right)+f_{k,\vec{x}_{il},\vec{y}_{jl}}\left(\vec{\Omega}_j^0\times\vec{y}_{jl}\right)\right]\cdot\left(\vec{\Omega}_i^0\times\vec{x}_{il}\right)$$
$$+\left[f_{k,\vec{y}_{il},\vec{x}_{jl}}\left(\vec{\Omega}_j^0\times\vec{x}_{jl}\right)+f_{k,\vec{y}_{il},\vec{y}_{jl}}\left(\vec{\Omega}_j^0\times\vec{y}_{jl}\right)\right]\cdot\left(\vec{\Omega}_i^0\times\vec{y}_{il}\right)$$
$$+\left[f_{k,\vec{x}_{jl},\vec{x}_{il}}\left(\vec{\Omega}_i^0\times\vec{x}_{il}\right)+f_{k,\vec{x}_{jl},\vec{y}_{il}}\left(\vec{\Omega}_i^0\times\vec{y}_{il}\right)\right]\cdot\left(\vec{\Omega}_j^0\times\vec{x}_{jl}\right)$$
$$+\left[f_{k,\vec{y}_{jl},\vec{x}_{il}}\left(\vec{\Omega}_i^0\times\vec{x}_{il}\right)+f_{k,\vec{y}_{jl},\vec{y}_{il}}\left(\vec{\Omega}_i^0\times\vec{y}_{il}\right)\right]\cdot\left(\vec{\Omega}_j^0\times\vec{y}_{jl}\right)=0\quad\text{for } k=1 \text{ to } m$$

Using the relation $\vec{a}\times\left(\vec{b}\times\vec{c}\right)=(\vec{a}\cdot\vec{c})\cdot\vec{b}-\left(\vec{a}\cdot\vec{b}\right)\cdot\vec{c}$ for transforming the double products, these equations become

$$\vec{f}_{k,\overrightarrow{OM}} \cdot \vec{J}^0(M) + \vec{f}_{k,\overrightarrow{ON}} \cdot \vec{J}^0(N) + f_{k,\overrightarrow{OM}^2}\left[\vec{V}^0(M)\right]^2 + f_{k,\overrightarrow{ON}^2}\left[\vec{V}^0(N)\right]^2$$
$$+ 2f_{k,\overrightarrow{OM}\,\overrightarrow{ON}} + \vec{V}^0(M) \cdot \vec{V}^0(N) + \vec{\gamma}_i^0 \cdot \left(\vec{x}_{il} \times \vec{f}_{k,\vec{x}_{il}} + \vec{y}_{il} \times \vec{f}_{k,\vec{y}_{il}}\right)$$
$$+ \vec{\gamma}_j^0 \cdot \left(\vec{x}_{jl} \times \vec{f}_{k,\vec{x}_{jl}} + \vec{y}_{jl} \times \vec{f}_{k,\vec{y}_{jl}}\right) - \vec{\Omega}_i^{0^2}\left(\vec{f}_{k,\vec{x}_{il}} \cdot \vec{x}_{il} + \vec{f}_{k,\vec{y}_{il}} \cdot \vec{y}_{il}\right)$$
$$- \vec{\Omega}_j^{0^2}\left(\vec{f}_{k,\vec{x}_{jl}} \cdot \vec{x}_{jl} + \vec{f}_{k,\vec{y}_{jl}} \cdot \vec{y}_{jl}\right) + 2f_{k,\vec{x}_{il},\vec{x}_{jl}}\left(\vec{\Omega}_j^0 \times \vec{x}_{jl}\right) \cdot \left(\vec{\Omega}_i^0 \times \vec{x}_{il}\right)$$
$$+ 2f_{k,\vec{x}_{il},\vec{y}_{jl}}\left(\vec{\Omega}_j^0 \times \vec{y}_{jl}\right) \cdot \left(\vec{\Omega}_i^0 \times \vec{x}_{il}\right) + 2f_{k,\vec{y}_{il},\vec{x}_{jl}}\left(\vec{\Omega}_j^0 \times \vec{x}_{jl}\right) \cdot \left(\vec{\Omega}_i^0 \times \vec{y}_{il}\right)$$
$$+ 2f_{k,\vec{y}_{il},\vec{y}_{jl}}\left(\vec{\Omega}_j^0 \times \vec{y}_{jl}\right) \cdot \left(\vec{\Omega}_i^0 \times \vec{y}_{il}\right)$$
$$+ 2\left[\vec{\Omega}_i^0 \times \left(f_{k,\overrightarrow{OM},\vec{x}_{il}}\vec{x}_{il} + f_{k,\overrightarrow{OM},\vec{y}_{il}}\vec{y}_{il}\right)\right.$$
$$\left. + \vec{\Omega}_j^0 \times \left(f_{k,\overrightarrow{OM},\vec{x}_{jl}}\vec{x}_{jl} + f_{k,\overrightarrow{OM},\vec{y}_{jl}}\vec{y}_{jl}\right)\right] \cdot \vec{V}^0(M)$$
$$+ 2\left[\vec{\Omega}_i^0 \times \left(f_{k,\overrightarrow{ON},\vec{x}_{il}}\vec{x}_{il} + f_{k,\overrightarrow{ON},\vec{y}_{il}}\vec{y}_{il}\right)\right.$$
$$\left. + \vec{\Omega}_j^0 \times \left(f_{k,\overrightarrow{ON},\vec{x}_{jl}}\vec{x}_{jl} + f_{k,\overrightarrow{ON},\vec{y}_{jl}}\vec{y}_{jl}\right)\right] \cdot \vec{V}^0(N) = 0$$
$$\text{for } k = 1 \text{ to } m$$

Noting also that $\vec{x}_{il} \cdot \vec{x}_{jl} = \vec{y}_{il} \cdot \vec{y}_{jl}$ and $\vec{x}_{il} \cdot \vec{y}_{jl} = -\vec{y}_{il} \cdot \vec{x}_{jl}$ and using the relation $\left(\vec{a} \times \vec{b}\right) \cdot \left(\vec{c} \times \vec{d}\right) = (\vec{a} \cdot \vec{c})\left(\vec{b} \cdot \vec{d}\right) - \left(\vec{a} \cdot \vec{d}\right)\left(\vec{c} \cdot \vec{b}\right)$ we finally obtain

$$\vec{f}_{k,\overrightarrow{OM}} \cdot \vec{J}^0(M) + \vec{f}_{k,\overrightarrow{ON}} \cdot \vec{J}^0(N) + f_{k,\overrightarrow{OM}^2}\left[\vec{V}^0(M)\right]^2 + f_{k,\overrightarrow{ON}^2}\left[\vec{V}^0(N)\right]^2$$
$$+ 2f_{k,\overrightarrow{OM}\,\overrightarrow{ON}}\vec{V}^0(M) \cdot \vec{V}^0(N) + \vec{\gamma}_i^0 \cdot \left(\vec{x}_{il} \times \vec{f}_{k,\vec{x}_{il}} + \vec{y}_{il} \times \vec{f}_{k,\vec{y}_{il}}\right)$$
$$+ \vec{\gamma}_j^0 \cdot \left(\vec{x}_{jl} \times \vec{f}_{k,\vec{x}_{jl}} + \vec{y}_{jl} \times \vec{f}_{k,\vec{y}_{jl}}\right) - \vec{\Omega}_i^{0^2}\left(\vec{f}_{k,\vec{x}_{il}} \cdot \vec{x}_{il} + \vec{f}_{k,\vec{y}_{il}} \cdot \vec{y}_{il}\right)$$
$$- \vec{\Omega}_j^{0^2}\left(\vec{f}_{k,\vec{x}_{jl}} \cdot \vec{x}_{jl} + \vec{f}_{k,\vec{y}_{jl}} \cdot \vec{y}_{jl}\right) + 2\left[\left(f_{k,\vec{x}_{il},\vec{x}_{jl}} + f_{k,\vec{y}_{il},\vec{y}_{jl}}\right)(\vec{x}_{il} \cdot \vec{x}_{jl})\right.$$
$$\left. + \left(f_{k,\vec{x}_{il},\vec{y}_{jl}} - f_{k,\vec{y}_{il},\vec{x}_{jl}}\right)(\vec{x}_{il} \cdot \vec{y}_{jl})\right]\left(\vec{\Omega}_i^0 \cdot \vec{\Omega}_j^0\right) + 2\left[\vec{\Omega}_i^0 \times \left(f_{k,\overrightarrow{OM},\vec{x}_{il}}\vec{x}_{il} + f_{k,\overrightarrow{OM},\vec{y}_{il}}\vec{y}_{il}\right)\right.$$
$$\left. + \vec{\Omega}_j^0 \times \left(f_{k,\overrightarrow{OM},\vec{x}_{jl}}\vec{x}_{jl} + f_{k,\overrightarrow{OM},\vec{y}_{jl}}\vec{y}_{jl}\right)\right] \cdot \vec{V}^0(M)$$
$$+ 2\left[\vec{\Omega}_i^0 \times \left(f_{k,\overrightarrow{ON},\vec{x}_{il}}\vec{x}_{il} + f_{k,\overrightarrow{ON},\vec{y}_{il}}\vec{y}_{il}\right)\right.$$
$$\left. + \vec{\Omega}_j^0 \times \left(f_{k,\overrightarrow{ON},\vec{x}_{jl}}\vec{x}_{jl} + f_{k,\overrightarrow{ON},\vec{y}_{jl}}\vec{y}_{jl}\right)\right] \cdot \vec{V}^0(N) = 0$$
$$\text{for } k = 1 \text{ to } m$$

which leads to Eq. (26).

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