微分几何课后标准答案（梅向明）

2 证明：
$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\boldsymbol{r}(t)}{\rho(t)}\right)=\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{\rho(t)}\boldsymbol{r}(t)\right)$$
$$=-\frac{\rho'(t)}{\rho^2(t)}\boldsymbol{r}(t)+\frac{1}{\rho(t)}\boldsymbol{r}'(t)$$
$$=\frac{\rho(t)\boldsymbol{r}'(t)-\rho'(t)\boldsymbol{r}(t)}{\rho^2(t)}.$$

3 证明：设 $\boldsymbol{r}(t)$在$[a,b]$上定义，且对于任一 $t\in(a,b)$有 $\boldsymbol{r}'(t)=\boldsymbol{0}$，则 $\boldsymbol{r}(t)$是$[a,b]$上的常向量. 因此在(a,b)上有任意阶微商，且都是 $\boldsymbol{0}$. 即
$$\boldsymbol{r}'(t)=\boldsymbol{r}''(t)=\cdots=\boldsymbol{r}^{(n)}(t)=\cdots=\boldsymbol{0},$$
于是 $\boldsymbol{r}(t+\Delta t)$有泰勒展开式：
$$\boldsymbol{r}(t+\Delta t)=\boldsymbol{r}(t)+\Delta t\boldsymbol{r}'(t)+\frac{1}{2!}(\Delta t)^2\boldsymbol{r}''(t)+\cdots+\frac{1}{n!}(\Delta t)^n\boldsymbol{r}^{(n)}(t)+\cdots$$
$$=\boldsymbol{r}(t)+\boldsymbol{0}\cdot\Delta t+\frac{1}{2!}\boldsymbol{0}\cdot(\Delta t)^2+\cdots+\frac{1}{n!}(\Delta t)^n\cdot\boldsymbol{0}+\cdots$$
$$=\boldsymbol{r}(t),$$
所以在 t 的邻域中 $\boldsymbol{r}(t)$是常向量. 考虑到 $t\in[a,b]$的任意性，则 $\boldsymbol{r}(t)$在$[a,b]$上是常向量.

4 证明：必要性　设 $\boldsymbol{r}(t)=\lambda(t)\boldsymbol{e}$（$\boldsymbol{e}$ 为常单位向量），则
$$\boldsymbol{r}'(t)=\lambda'(t)\boldsymbol{e},$$
所以
$$\boldsymbol{r}(t)\times\boldsymbol{r}'(t)=\boldsymbol{0}.$$

充分性　设 $\boldsymbol{r}(t)=\lambda(t)\boldsymbol{e}(t)$（$\boldsymbol{e}(t)$为单位向量函数），则
$$\boldsymbol{r}'(t)=\lambda'(t)\boldsymbol{e}(t)+\lambda(t)\boldsymbol{e}'(t),$$
$$\boldsymbol{r}(t)\times\boldsymbol{r}'(t)=\lambda^2(t)[\boldsymbol{e}(t)\times\boldsymbol{e}'(t)].$$
因为 $\boldsymbol{r}(t)\neq\boldsymbol{0}$，于是 $\lambda(t)\neq0$ 当 $\boldsymbol{r}(t)\times\boldsymbol{r}'(t)=\boldsymbol{0}$，从而有
$$\boldsymbol{e}(t)\times\boldsymbol{e}'(t)=\boldsymbol{0},$$
即 $\boldsymbol{e}(t)/\!/\boldsymbol{e}'(t)$，因为 $\boldsymbol{e}(t)\perp\boldsymbol{e}'(t)$（根据$|\boldsymbol{e}(t)|=1$），因此 $\boldsymbol{e}'(t)=\boldsymbol{0}$，即 $\boldsymbol{e}(t)$为常向量，所以
$$\boldsymbol{r}(t)=\lambda(t)\boldsymbol{e}(t)$$
有固定方向.

5　证明：必要性　设固定平面 π 的单位法向量为 $\boldsymbol{n}$. 依题意 $\boldsymbol{r}(t)\perp\boldsymbol{n}$，则 $\boldsymbol{r}(t)\cdot\boldsymbol{n}=0$. 从而

$$\boldsymbol{r}'(t)\cdot\boldsymbol{n}=0,$$
$$\boldsymbol{r}''(t)\cdot\boldsymbol{n}=0.$$

$\boldsymbol{r}(t),\boldsymbol{r}'(t),\boldsymbol{r}''(t)$ 均与 $\boldsymbol{n}$ 垂直，所以 $(\boldsymbol{r}(t),\boldsymbol{r}'(t),\boldsymbol{r}''(t))=0$.

充分性　由已知，$\boldsymbol{r}(t),\boldsymbol{r}'(t),\boldsymbol{r}''(t)$ 共面. 若

$$\boldsymbol{r}(t)\times\boldsymbol{r}'(t)\equiv\boldsymbol{0},$$

则由 $\boldsymbol{r}(t)\neq\boldsymbol{0}$ 可知 $\boldsymbol{r}(t)$ 有固定方向（上题），所以 $\boldsymbol{r}(t)$ 平行于固定平面.

若　$\boldsymbol{r}(t)\times\boldsymbol{r}''(t)\neq\boldsymbol{0}$，则由 $\boldsymbol{r}(t),\boldsymbol{r}'(t),\boldsymbol{r}''(t)$ 共面可知

$$\boldsymbol{r}''(t)=\lambda(t)\boldsymbol{r}(t)+\mu(t)\boldsymbol{r}'(t),$$

记 $\boldsymbol{n}(t)=\boldsymbol{r}(t)\times\boldsymbol{r}'(t)$，则

$$\boldsymbol{n}'(t)=\boldsymbol{r}(t)\times\boldsymbol{r}''(t)=\mu(t)\boldsymbol{r}(t)\times\boldsymbol{r}'(t)=\mu(t)\boldsymbol{n}(t).$$

从而有

$$\boldsymbol{n}(t)\times\boldsymbol{n}'(t)=\boldsymbol{0},\text{但}\quad \boldsymbol{n}(t)\neq\boldsymbol{0}.$$

因此 $\boldsymbol{n}(t)$ 有固定方向（上题）. 又 $\boldsymbol{r}(t)\perp\boldsymbol{n}(t)$，所以 $\boldsymbol{r}(t)$ 平行于固定平面.

习题　1.2

1　由 $\boldsymbol{r}'(t)=\{-\sin t,\cos t,1\}$ 得

$$|\boldsymbol{r}'(t)|=\sqrt{(-\sin t)^2+(\cos t)^2+1}=\sqrt{2}\neq 0,$$

所以曲线是正则曲线.

令 $\begin{cases}\cos t=1,\\ \sin t=0,\end{cases}$ 解出 $t=0$，则对应于点 $(1,0,0)$ 有 $t=0$，所以

$$\boldsymbol{r}'(0)=\{0,1,1\},$$

则曲线在 $(1,0,0)$ 点（即 $t=0$ 点）的切线方程为

$$\frac{x-1}{0}=\frac{y-0}{1}=\frac{z-0}{1},$$

或

$$\boldsymbol{\rho}=\boldsymbol{e}_1+\lambda\boldsymbol{e}_2+\lambda\boldsymbol{e}_3.$$

法面方程为

$$(\boldsymbol{\rho}-\boldsymbol{r}(0))\cdot\boldsymbol{r}_t'(0)=0,$$

即

$$(x-1)0+(y-0)1+(z-0)1=0,$$
$$y+z=0.$$

2 $$\boldsymbol{r}(t)=\{at,bt^2,ct^3\},$$
$$\boldsymbol{r}'(t)=\{a,2bt,3ct^2\}.$$

所以,切线方程为

$$\boldsymbol{\rho}-\boldsymbol{r}(t_0)=\lambda\boldsymbol{r}'(t_0),$$

即 $$\frac{x-at_0}{a}=\frac{y-bt_0^2}{2bt_0}=\frac{z-ct_0^3}{3ct_0^2}.$$

法面方程为

$$[\boldsymbol{\rho}-\boldsymbol{r}(t_0)]\cdot\boldsymbol{r}'(t_0)=0,$$

即 $$(x-at_0)\cdot a+(y-bt_0^2)\cdot 2bt_0+(z-ct_0^3)\cdot 3ct_0^2=0,$$
$$ax+2bt_0y+3ct_0^2z-(a^2t_0+2b^2t_0^3+3c^2t_0^5)=0,$$

5 因为 $\boldsymbol{r}'(\theta)=\{-a\sin\theta,a\cos\theta,b\}$,取 z 轴上的单位向量 $\boldsymbol{e}_3=\{0,0,1\}$,则

$$\cos(\widehat{\boldsymbol{r}',\boldsymbol{e}_3})=\frac{\boldsymbol{r}'\cdot\boldsymbol{e}_3}{|\boldsymbol{r}'||\boldsymbol{e}_3|}$$
$$=\frac{-a\sin\theta\cdot 0+a\cos\theta\cdot 0+b\cdot 1}{\sqrt{(-a\sin\theta)^2+(a\cos\theta)^2+b^2\cdot 1}}$$

$$=\frac{b}{\sqrt{a^2+b^2}}=\text{常数}.$$

即 $\boldsymbol{r}'$ 与 $\boldsymbol{e}_3$ 的夹角不随 θ 的变化而变化,因之曲线的切线与 z 轴作固定角.

12 $\boldsymbol{r}=\{t,a\cosh\frac{t}{a},0\}$,

$$\boldsymbol{r}'=\{1,\sinh\frac{t}{a},0\},|\boldsymbol{r}'|=\sqrt{1+\sinh^2\frac{t}{a}}=\cosh\frac{t}{a}.$$

∴ 从 $t=0$ 算起的弧长为:

$$l(t)=\int_0^t|\boldsymbol{r}'|\mathrm{d}t$$
$$=\int_0^t\cosh\frac{t}{a}\mathrm{d}t$$
$$=a\int_0^{\frac{t}{a}}\cosh u\,\mathrm{d}u$$
$$=a\sinh\frac{t}{a}.$$

13 ∵ 曲线(C)的方程为 $y=bx^2$，它的向量参数表示为：

$$\boldsymbol{r}=\{x,bx^2,0\},$$

$$\boldsymbol{r}'=\{1,2bx,0\},|\boldsymbol{r}'|=\sqrt{1+4b^2x^2}.$$

对应于 $-a\leqslant x\leqslant a$ 一段的弧长为：

$$\begin{aligned}l(x)&=\int_{-a}^{a}\sqrt{1+4b^2x^2}\,\mathrm{d}x\\&=2\int_0^a\sqrt{1+4b^2x^2}\,\mathrm{d}x\\&=\frac{1}{b}\int_0^{2ab}\sqrt{1+u^2}\,\mathrm{d}u\\&=\frac{1}{b}\left[\frac{u}{2}\sqrt{1+u^2}+\frac{1}{2}\ln(u+\sqrt{1+u^2})\right]_0^{2ab}.\\&=a\sqrt{1+4a^2b^2}+\frac{1}{2b}\ln\left(2ab+\sqrt{1+4a^2b^2}\right).\end{aligned}$$

14 $\boldsymbol{r}=\{a\cos^3t,a\sin^3t,0\}$,

$\boldsymbol{r}'=\{-3a\cos^2t\sin t,3a\sin^2t\cos t,0\}$,

$|\boldsymbol{r}'|=|3a\sin t\cos t|=3a|\sin t\cos t|$.

$0\leqslant t\leqslant\frac{\pi}{2}$ 一段的弧长为：

$$l(t)=\int_0^{\frac{\pi}{2}}3a\sin t\cos t\,\mathrm{d}t$$

$$= 3a\int_0^{\frac{\pi}{2}} \sin t\,\mathrm{d}(\sin t)$$

$$= \frac{3}{2}a\sin^2 t\Big|_0^{\frac{\pi}{2}}$$

$$= \frac{3}{2}a.$$

15 $\boldsymbol{r} = \{a(t-\sin t), a(1-\cos t), 0\}, a>0,$

$$\boldsymbol{r}' = \{a(1-\cos t), a\sin t, 0\},$$

$$|\boldsymbol{r}'| = \sqrt{[a(1-\cos t)]^2 + (a\sin t)^2}$$

$$= 2a\left|\sin\frac{t}{2}\right|.$$

对应 $0\leqslant t\leqslant 2\pi$ 一段的弧长为：

$$l = \int_0^{2\pi} 2a\left|\sin\frac{t}{2}\right|\mathrm{d}t$$

$$= 4a\int_0^{\pi}\sin u\,\mathrm{d}u$$

$$= 8a.$$

16 曲线与 xOy 平面相交时，$z=0$，即 $4at=0$，得 $t=0$.

$$\boldsymbol{r} = \{3a\cos t, 3a\sin t, 4at\},$$

$$\boldsymbol{r}' = \{-3a\sin t, 3a\cos t, 4a\},$$

$$|\boldsymbol{r}'| = \sqrt{9a^2+16a^2} = 5a.$$

所以，弧长

$$l(t) = \int_0^t 5a\,\mathrm{d}t = 5at.$$

17 曲线与两平面交点的横坐标分别为 $x=a$，$x=3a$. 取 x 为参数，曲线的方程为

$$\boldsymbol{r} = \left\{x, \frac{x^3}{3a^2}, \frac{a^2}{2x}\right\},$$

$$\boldsymbol{r}'_x = \left\{1, \frac{x^2}{a^2}, -\frac{a^2}{2x^2}\right\},$$

$$|\boldsymbol{r}'| = \frac{\sqrt{(2a^2x^2)^2 + (2x^4)^2 + (a^4)^2}}{2a^2x^2} = \frac{2x^4 + a^4}{2a^2x^2}.$$

所以 $l = \int_a^{3a} \frac{2x^4 + a^4}{2a^2x^2} \mathrm{d}x$

$$= \int_a^{3a} \left(\frac{x^2}{a^2} + \frac{a^2}{2x^2} \right) \mathrm{d}x = 9a.$$

24 $\boldsymbol{r} = \{a\cos t, a\sin t, bt\}$,

$\boldsymbol{r}' = \{-a\sin t, a\cos t, b\}$,

$|\boldsymbol{r}'| = \sqrt{a^2 + b^2}$,

$s(t) = \int_0^t \sqrt{a^2 + b^2} \mathrm{d}t = \sqrt{a^2 + b^2}\, t$,

则 $t = \frac{s}{\sqrt{a^2 + b^2}}$,代入原方程,得

$$\boldsymbol{r} = \left\{ a\cos \frac{s}{\sqrt{a^2 + b^2}}, a\sin \frac{s}{\sqrt{a^2 + b^2}}, b \frac{s}{\sqrt{a^2 + b^2}} \right\}.$$

27 由于 $x = x(\theta) = \rho(\theta)\cos\theta$,

$y = y(\theta) = \rho(\theta)\sin\theta$.

所以 $\boldsymbol{r} = \{\rho(\theta)\cos\theta, \rho(\theta)\sin\theta\}$,

$\boldsymbol{r}' = \{\rho'\cos\theta - \rho\sin\theta, \rho'\sin\theta + \rho\cos\theta\}$,

$$|\boldsymbol{r}'| = \sqrt{(\rho'\cos\theta - \rho\sin\theta)^2 + (\rho'\sin\theta + \rho\cos\theta)^2}$$

$$= \sqrt{(\rho')^2 + \rho^2}.$$

$\therefore$ $l(\theta) = \int_{\theta_0}^{\theta} |\boldsymbol{r}'| \mathrm{d}\theta = \int_{\theta_0}^{\theta} \sqrt{(\rho')^2 + \rho^2} \mathrm{d}\theta.$

习题 1.3

1 解：$\boldsymbol{r}(t)=\{a\cos t, a\sin t, bt\}$,

$\boldsymbol{r}'(t)=\{-a\sin t, a\cos t, b\}$,

$\boldsymbol{r}''(t)=\{-a\cos t, -a\sin t, 0\}$,

密切平面的方程为：

$$(\boldsymbol{R}-\boldsymbol{r}, \boldsymbol{r}', \boldsymbol{r}'')=0,$$

即

$$\begin{vmatrix} X-a\cos t & Y-a\sin t & Z-bt \\ -a\sin t & a\cos t & b \\ -a\cos t & -a\sin t & 0 \end{vmatrix}=0.$$

展开整理得：

$$X\sin t-Y\cos t+\frac{a}{b}Z-at=0.$$

2 解：$\boldsymbol{r}=\{t\sin t, t\cos t, te^t\}$,

$\boldsymbol{r}'(t)=\{t\cos t+\sin t, \cos t-t\sin t, e^t+te^t\}$,

$\boldsymbol{r}''(t)=\{2\cos t-t\sin t, -2\sin t-t\cos t, 2e^t+te^t\}$.

在原点处 $t=0$,

$$\boldsymbol{r}(0)=\{0,0,0\},$$
$$\boldsymbol{r}'(0)=\{0,1,1\},$$
$$\boldsymbol{r}''(0)=\{2,0,2\}.$$

在原点处切平面的方程为：

$$(\boldsymbol{R}-\boldsymbol{r}_0, \boldsymbol{r}'_0, \boldsymbol{r}''_0)=0.$$

即

$$X+Y-Z=0.$$

法平面的方程为：

$$(\boldsymbol{R}-\boldsymbol{r}_0)\cdot\boldsymbol{r}'_0=0,$$

即

$$Y+Z=0.$$

从切平面的方程为：

$$(\boldsymbol{R}-\boldsymbol{r}_0,\boldsymbol{r}'_0\times\boldsymbol{r}''_0,\boldsymbol{r}'_0)=0,\boldsymbol{r}'_0\times\boldsymbol{r}''_0=\{1,1,-1\},$$

即

$$2X-Y+Z=0.$$

切线方程为：

$$\boldsymbol{R}-\boldsymbol{r}_0=\lambda\boldsymbol{r}'_0,$$

即

$$\frac{X}{0}=\frac{Y}{1}=\frac{Z}{1}.$$

主法线方程为：

$$\boldsymbol{R}-\boldsymbol{r}_0=\lambda[(\boldsymbol{r}'_0\times\boldsymbol{r}''_0)\times\boldsymbol{r}'_0],$$

由于

$$(\boldsymbol{r}'_0\times\boldsymbol{r}''_0)\times\boldsymbol{r}'_0=\{2,-1,1\},$$

主法线方程为：

$$\frac{X}{2}=\frac{Y}{-1}=\frac{Z}{1}.$$

副法线方程为

$$(\boldsymbol{R}-\boldsymbol{r}_0)=\lambda(\boldsymbol{r}'_0\times\boldsymbol{r}''_0),$$

即

$$\frac{X}{1}=\frac{Y}{1}=\frac{Z}{-1}.$$

3 $\boldsymbol{r}=\{a\cos t,a\sin t,bt\}$,

$\boldsymbol{r}'=\{-a\sin t,a\cos t,b\}$,

$\boldsymbol{r}''=\{-a\cos t,-a\sin t,0\}$,

$\boldsymbol{r}'\times\boldsymbol{r}''=\{ab\sin t,-ab\cos t,a^2\}$,

$(\boldsymbol{r}'\times\boldsymbol{r}'')\times\boldsymbol{r}'=\{-(ab^2+a^3)\cos t,-(a^3+ab^2)\sin t,0\}$,

$|(\boldsymbol{r}'\times\boldsymbol{r}'')\times\boldsymbol{r}'|=ab^2+a^3$,

$\boldsymbol{\beta}=\dfrac{(\boldsymbol{r}'\times\boldsymbol{r}'')\times\boldsymbol{r}'}{|(\boldsymbol{r}'\times\boldsymbol{r}'')\times\boldsymbol{r}'|}=\{-\cos t,-\sin t,0\}$,

主法线的方程为：

$$(\boldsymbol{R}-\boldsymbol{r})=\lambda\boldsymbol{\beta},$$

即
$$\frac{X-a\cos t}{-\cos t}=\frac{Y-a\sin t}{-\sin t}=\frac{Z-bt}{0}.$$

又 z 轴的方程为：

$$\frac{X}{0}=\frac{Y}{0}=\frac{Z}{1},$$

对任意 t，有

$$\boldsymbol{\beta}\cdot\boldsymbol{e}_3=-\cos t\cdot 0+(-\sin t)\cdot 0+0\cdot 1=0,$$

即主法线与 z 轴垂直. 又由于点 $(0,0,bt)$ 即在主法线上，又在 z 轴上，故主法线与 z 轴垂直相交于 $(0,0,bt)$.

4 解：$\boldsymbol{r}=\{\cos\alpha\cos t,\cos\alpha\sin t,t\sin\alpha\}$,

$\boldsymbol{r}'=\{-\cos\alpha\sin t,\cos\alpha\cos t,\sin\alpha\}$,

$\boldsymbol{r}''=\{-\cos\alpha\cos t,-\cos\alpha\sin t,0\}$,

$\boldsymbol{r}'\times\boldsymbol{r}''=\{\sin\alpha\cos\alpha\sin t,-\sin\alpha\cos\alpha\cos t,\cos^2\alpha\}$,

$|\boldsymbol{r}'\times\boldsymbol{r}''|=\cos\alpha$,

所以 $\quad\boldsymbol{\gamma}=\{\sin\alpha\sin t,-\sin\alpha\cos t,\cos\alpha\}$.

新曲线的方程为：

$$\begin{aligned}\tilde{\boldsymbol{r}}&=\boldsymbol{r}+\boldsymbol{\gamma}\\&=\{\cos\alpha\cos t+\sin\alpha\sin t,\cos\alpha\sin t-\sin\alpha\cos t,\\&\quad t\sin\alpha+\cos\alpha\}\\&=\{\cos(t-\alpha),\sin(t-\alpha),t\sin\alpha+\cos\alpha\},\end{aligned}$$

$$\tilde{\boldsymbol{r}}'=\{-\sin(t-\alpha),\cos(t-\alpha),\sin\alpha\},$$

$$\tilde{\boldsymbol{r}}''=\{-\cos(t-\alpha),-\sin(t-\alpha),0\},$$

新曲线密切平面的方程为

$$\begin{vmatrix}X-\cos(t-\alpha) & Y-\sin(t-\alpha) & Z-(t\sin\alpha+\cos\alpha)\\-\sin(t-\alpha) & \cos(t-\alpha) & \sin\alpha\\-\cos(t-\alpha) & -\sin(t-\alpha) & 0\end{vmatrix}=0.$$

展开整理得

$$[\sin\alpha\sin(t-\alpha)]X-[\sin\alpha\cos(t-\alpha)]Y+$$
$$Z-(t\sin\alpha+\cos\alpha)=0.$$

5 证明：设球面的半径为 R，球心在原点，球面曲线的方程为

$$\boldsymbol{r}=\boldsymbol{r}(s),$$
$$\boldsymbol{r}^2=R^2,$$

则

$$\boldsymbol{r}\cdot\dot{\boldsymbol{r}}=0.$$

曲线的法平面方程为

$$[\boldsymbol{\rho}-\boldsymbol{r}(s)]\cdot\dot{\boldsymbol{r}}(s)=0,$$

即

$$\boldsymbol{\rho}(s)\cdot\dot{\boldsymbol{r}}(s)=0.$$

它通过原点，即通过球心.

6 证明：因为 $\boldsymbol{r}=\{a\cos t,a\sin t,bt\}$,

$$\boldsymbol{r}'=\{-a\sin t,a\cos t,b\},$$
$$\boldsymbol{r}''=\{-a\cos t,-a\sin t,0\},$$
$$\boldsymbol{r}'\times\boldsymbol{r}''=\{ab\sin t,-ab\cos t,a^2\}.$$

所以副法线的方向向量为 $\{b\sin t,-b\cos t,a\}$，过原点且平行于副法线的直线方程为

$$\frac{x}{b\sin t}=\frac{y}{-b\cos t}=\frac{z}{a}.$$

消去 t：

$$x=\lambda b\sin t,y=-\lambda b\cos t,z=a\lambda,$$
$$x^2+y^2=b^2\lambda^2,z^2=a^2\lambda^2,$$

即得

$$a^2(x^2+y^2)=b^2z^2.$$

7 (1) 因为 $\boldsymbol{r}=\{a\cosh t,a\sinh t,at\}$,

$$\boldsymbol{r}'=\{a\sinh t,a\cosh t,a\},$$
$$\boldsymbol{r}''=\{a\cosh t,a\sinh t,0\},$$
$$\boldsymbol{r}'\times\boldsymbol{r}''=\{-a^2\sinh t,a^2\cosh t,-a^2\},$$

$|\boldsymbol{r}'|=\sqrt{2}a\cosh t,$

$|\boldsymbol{r}'\times\boldsymbol{r}''|=\sqrt{2}a^2\cosh t,$

$\boldsymbol{r}'''=\{a\sinh t, a\cosh t, 0\},$

$(\boldsymbol{r}',\boldsymbol{r}'',\boldsymbol{r}''')=a^3,$

所以 $$k=\frac{1}{2a\cosh^2 t},\ \tau=\frac{1}{2a\cosh^2 t}.$$

(2) 因为 $\boldsymbol{r}=\{a(3t-t^3), 3at^2, a(3t+t^3)\},$

$\boldsymbol{r}'=\{3a(1-t^2), 6at, 3a(1+t^2)\},$

$\boldsymbol{r}''=\{-6at, 6a, 6at\},$

$\boldsymbol{r}'\times\boldsymbol{r}''=\{18a^2(t^2-1), -36a^2t, 18a^2(t^2+1)\},$

$|\boldsymbol{r}'|=3\sqrt{2}a(1+t^2),$

$|\boldsymbol{r}'\times\boldsymbol{r}''|=18\sqrt{2}a^2(1+t^2),$

$\boldsymbol{r}'''=\{-6a, 0, 6a\},$

$(\boldsymbol{r}',\boldsymbol{r}'',\boldsymbol{r}''')=216a^3,$

所以

$$k=\frac{1}{3a(1+t^2)^2},\ \tau=\frac{1}{3a(1+t^2)^2}.$$

8 解:因为 $\boldsymbol{r}=\{\cos^3 t, \sin^3 t, \cos^2 t\},$

$\boldsymbol{r}'=\{-3\cos t, 3\sin t, -2\}\sin t\cos t,$

$\boldsymbol{r}''=\{3\cos t(3\sin^2 t-1), 3\sin t(3\cos^2 t-1), -4\cos 2t\},$

$\boldsymbol{r}'\times\boldsymbol{r}''=\sin^2 2t\left\{\cos t, -\sin t, -\frac{3}{4}\right\},$

$|\boldsymbol{r}'|=5|\sin t\cos t|,$

$|\boldsymbol{r}'\times\boldsymbol{r}''|=\frac{15}{4}\sin^2 2t=15\sin^2 t+\cos^2 t,$

$\boldsymbol{r}'''=\{3\sin t(9\cos^2 t-2), 3\cos t(2-9\sin^2 t), 8\sin 2t\},$

$(\boldsymbol{r}',\boldsymbol{r}'',\boldsymbol{r}''')=36\sin^3 t\cos^3 t.$

所以,曲率 k,挠率 τ 分别为

$$k=\frac{\boldsymbol{r}'\times\boldsymbol{r}''}{|\boldsymbol{r}'|^3}=\frac{15\sin^2 t+\cos^2 t}{(5|\sin t\cos t|)^3}=\frac{3}{25|\sin t\cos t|},$$

(1) 当 $0<t<\frac{\pi}{2}, \pi<t<\frac{3}{2}\pi$ 时,

$$|\sin t\cos t| = \sin t\cos t.$$

这时

$$\boldsymbol{\alpha} = \left\{-\frac{3}{5}\cos t, \frac{3}{5}\sin t, -\frac{4}{5}\right\},$$

$$\boldsymbol{\beta} = \{\sin t, \cos t, 0\},$$

$$\boldsymbol{\gamma} = \left\{\frac{4}{5}\cos t, -\frac{4}{5}\sin t, -\frac{3}{5}\right\}.$$

(2) 当 $\frac{\pi}{2}<t<\pi, \frac{3}{2}\pi<t<2\pi$ 时.

$$|\sin t\cos t| = -\sin t\cos t,$$

这时

$$\boldsymbol{\alpha} = -\left\{-\frac{3}{5}\cos t, \frac{3}{5}\sin t - \frac{4}{5}\right\} = \left\{\frac{3}{5}\cos t, -\frac{3}{5}\sin t, \frac{4}{5}\right\},$$

$$\tau = \frac{(\boldsymbol{r}', \boldsymbol{r}'', \boldsymbol{r}''')}{|\boldsymbol{r}'\times\boldsymbol{r}''|^2} = \frac{36\sin^3 t\cos^3 t}{(15\sin^2 t\cos^2 t)^2} = \frac{4}{25\sin t\cos t},$$

$$\boldsymbol{\alpha} = \frac{\boldsymbol{r}'}{|\boldsymbol{r}'|} = \frac{\sin t\cos t}{5|\sin t\cos t|}\{-3\cos t, 3\sin t, -4\}$$

$$= \frac{\sin t\cos t}{|\sin t\cos t|}\left\{-\frac{3}{5}\cos t, \frac{3}{5}\sin t, -\frac{4}{5}\right\},$$

$$\boldsymbol{\gamma} = \frac{\boldsymbol{r}'\times\boldsymbol{r}''}{|\boldsymbol{r}'\times\boldsymbol{r}''|} = \frac{3\sin^2 2t}{\frac{15}{4}\sin^2 2t}\left\{\cos t, -\sin t, -\frac{3}{4}\right\}$$

$$= \left\{\frac{4}{5}\cos t, -\frac{4}{5}\sin t, -\frac{3}{5}\right\},$$

$$\boldsymbol{\beta} = \boldsymbol{\gamma}\times\boldsymbol{\alpha} = \frac{\sin t\cos t}{|\sin t\cos t|}\{\sin t, \cos t, 0\}.$$

根据 $\sin t, \cos t$ 的周期性,所有讨论只考虑 $0\leqslant t\leqslant 2\pi$ 即可. 当 $t=0$、$\frac{\pi}{2}$、π、$\frac{3}{2}\pi$、2π 时,在对应点 $\boldsymbol{r}'=\boldsymbol{0}$,即这些点是曲线的非正常点.

$$\boldsymbol{\beta}=-\{\sin t,\cos t,0\}=\{-\sin t,-\cos t,0\},$$

$$\boldsymbol{\gamma}=\left\{\frac{4}{5}\cos t,-\frac{4}{5}\sin t,0\right\}.$$

下面验证伏雷内公式：

由于 $\frac{\mathrm{d}s}{\mathrm{d}t}=|\boldsymbol{r}'|=5|\sin t\cos t|$，当 $0<t<\frac{\pi}{2}$、$\pi<t<\frac{3}{2}\pi$ 时，由于

$$|\sin t\cos t|=\sin t\cos t,$$

(1)
$$\dot{\boldsymbol{\alpha}}=\frac{\mathrm{d}\boldsymbol{\alpha}}{\mathrm{d}s}=\frac{\frac{\mathrm{d}\boldsymbol{\alpha}}{\mathrm{d}t}}{\frac{\mathrm{d}s}{\mathrm{d}t}}=\frac{1}{|\boldsymbol{r}'|}\cdot\frac{\mathrm{d}\boldsymbol{\alpha}}{\mathrm{d}t}$$

$$=\frac{1}{5|\sin t\cos t|}\left\{\frac{3}{5}\sin t,\frac{3}{5}\cos t,0\right\},$$

$$=\left\{\frac{3}{25\cos t},\frac{3}{25\sin t},0\right\},$$

$$k\boldsymbol{\beta}=\frac{3}{25|\sin t\cos t|}\{\sin t,\cos t,0\}$$

$$=\left\{\frac{3}{25\cos t},\frac{3}{25\sin t},0\right\},$$

即
$$\dot{\boldsymbol{\alpha}}=k\boldsymbol{\beta}.$$

(2)
$$\dot{\boldsymbol{\beta}}=\frac{\mathrm{d}\boldsymbol{\beta}}{\mathrm{d}s}=\frac{\frac{\mathrm{d}\boldsymbol{\beta}}{\mathrm{d}t}}{\frac{\mathrm{d}s}{\mathrm{d}t}}=\frac{1}{|\boldsymbol{r}'|}\cdot\frac{\mathrm{d}\boldsymbol{\beta}}{\mathrm{d}t}$$

$$=\frac{1}{5|\sin t\cos t|}\{\cos t,-\sin t,0\}$$

$$=\left\{\frac{1}{5\sin t},-\frac{1}{5\cos t},0\right\},$$

$$-k\boldsymbol{\alpha}+\tau\boldsymbol{\gamma}=-\frac{3}{25|\sin t\cos t|}\left\{-\frac{3}{5}\cos t,\frac{3}{5}\sin t,-\frac{4}{5}\right\}+$$

$$\frac{4}{25\sin t\cos t}\left\{\frac{4}{5}\cos t,-\frac{4}{5}\sin t,-\frac{3}{5}\right\}$$

$$= \left\{\frac{1}{5\sin t}, -\frac{1}{5\cos t}, 0\right\},$$

即 $$\dot{\boldsymbol{\beta}} = -k\boldsymbol{\alpha} + \tau\boldsymbol{\gamma}.$$

(3) $$\dot{\boldsymbol{\gamma}} = \frac{\mathrm{d}\boldsymbol{\gamma}}{\mathrm{d}s} = \frac{\frac{\mathrm{d}\boldsymbol{\gamma}}{\mathrm{d}t}}{\frac{\mathrm{d}s}{\mathrm{d}t}} = \frac{1}{|\boldsymbol{r}'|}\cdot\frac{\mathrm{d}\boldsymbol{\gamma}}{\mathrm{d}t}$$

$$= \frac{1}{|5\sin t\cos t|}\left\{-\frac{4}{5}\sin t, -\frac{4}{5}\cos t, 0\right\}$$

$$= \left\{-\frac{4}{25\cos t}, -\frac{4}{25\sin t}, 0\right\}.$$

$$-\tau\boldsymbol{\beta} = -\frac{4}{25\sin t\cos t}\{\sin t, \cos t, 0\}$$

$$= \left\{-\frac{4}{25\cos t}, -\frac{4}{25\sin t}, 0\right\},$$

即 $\dot{\boldsymbol{\gamma}} = -\tau\boldsymbol{\beta}$.

对于$\frac{\pi}{2}<t<\pi$，$\frac{3}{2}\pi<t<2\pi$时，完全可以按上述方法验证.

9 证法一，设所给曲线为(C)：$\boldsymbol{r}=\boldsymbol{r}(s)$，定点的向径为 $\boldsymbol{R}_0$，则

$$\boldsymbol{r}(s)-\boldsymbol{R}_0=\lambda(s)\boldsymbol{\alpha}(s),$$

$$\boldsymbol{\alpha}(s)=\dot{\lambda}(s)\boldsymbol{\alpha}+\lambda k\boldsymbol{\beta}.$$

但 $\boldsymbol{\alpha}$，$\boldsymbol{\beta}$ 线性无关，从而

$$\dot{\lambda}=1, \lambda k=0,$$

又 $\lambda\neq0$，所以 $k=0$，即(C)是直线.

证法二　根据已知，有

$$[\boldsymbol{r}(s)-\boldsymbol{R}_0]\times\boldsymbol{\alpha}(s)=\boldsymbol{0},$$

$$\dot{\boldsymbol{r}}(s)\times\boldsymbol{\alpha}+[\boldsymbol{r}(s)-\boldsymbol{R}_0]\times k\boldsymbol{\beta}=0,$$

$$[\boldsymbol{r}(s)-\boldsymbol{R}_0]\times k\boldsymbol{\beta}=0,$$

但

$$[\boldsymbol{r}(s)-\boldsymbol{R}_0]\times\boldsymbol{\beta}\neq0.$$

(否则，$(\boldsymbol{r}(s)-\boldsymbol{R}_0)\parallel\boldsymbol{\beta}$，由已知得出 $(\boldsymbol{r}(s)-\boldsymbol{R}_0)\parallel\boldsymbol{\alpha}$，于是 $(\boldsymbol{r}(s)-\boldsymbol{R}_0)\equiv 0$，即 $\boldsymbol{r}(s)\equiv\boldsymbol{R}_0$，从而所给的曲线退缩为一点，得出矛盾)，所以

$$k\equiv 0$$

即曲线(C)是直线.

证法三　设所给曲线为(C)：$\boldsymbol{r}=\boldsymbol{r}(t)$，则由已知有

$$\boldsymbol{r}(t)-\boldsymbol{R}_0=\lambda(t)\boldsymbol{r}'(t),$$
$$\boldsymbol{r}'(t)=\lambda'(t)\boldsymbol{r}'(t)+\lambda(t)\boldsymbol{r}''(t).$$

于是　$\boldsymbol{r}'\times\boldsymbol{r}''=0$，所以

$$k=\frac{|\boldsymbol{r}'\times\boldsymbol{r}''|}{|\boldsymbol{r}'|^3}=0,$$

即曲线(C)是直线.

10　证法一　设曲线(C)：$\boldsymbol{r}=\boldsymbol{r}(t)$，定点向径为 $\boldsymbol{R}_0$，据已知条件$(\boldsymbol{r}(t)-\boldsymbol{R}_0)$在密切平面上，故

$$(\boldsymbol{r}(t)-\boldsymbol{R}_0,\boldsymbol{r}',\boldsymbol{r}'')=0. \qquad (*)$$

(1) 若 $\boldsymbol{r}-\boldsymbol{R}_0,\boldsymbol{r}',\boldsymbol{r}''$有两个共线，则分别有下列结果：

① 若$(\boldsymbol{r}-\boldsymbol{R}_0)\parallel\boldsymbol{r}'$则据上题结论，$(C)$是直线；

② 或 $\boldsymbol{r}'\parallel\boldsymbol{r}''$，则 $\boldsymbol{r}'\times\boldsymbol{r}''=0$　$\therefore k=0$，曲线(C)是直线；

③ 若$(\boldsymbol{r}-\boldsymbol{R}_0)\parallel\boldsymbol{r}''$，设 $\boldsymbol{r}-\boldsymbol{R}_0=\lambda(t)\boldsymbol{r}''$，两边对 t 求微商：

$$\boldsymbol{r}'=\lambda'(t)\boldsymbol{r}''+\lambda(t)\boldsymbol{r}''',$$

即　$\boldsymbol{r}',\boldsymbol{r}'',\boldsymbol{r}'''$共面. 故$(\boldsymbol{r}',\boldsymbol{r}'',\boldsymbol{r}''')=0$. 故

$$\tau=\frac{(\boldsymbol{r}',\boldsymbol{r}'',\boldsymbol{r}''')}{|\boldsymbol{r}'\times\boldsymbol{r}''|^2}=0,$$

则(C)是平面曲线.

(2) 若 $\boldsymbol{r}-\boldsymbol{R}_0,\boldsymbol{r}',\boldsymbol{r}''$两两不共线，则在$(*)$式两边对 t 求微商：

$$(\boldsymbol{r}-\boldsymbol{R}_0,\boldsymbol{r}',\boldsymbol{r}'')'=0,$$

即　$(\boldsymbol{r}',\boldsymbol{r}',\boldsymbol{r}'')+(\boldsymbol{r}-\boldsymbol{R}_0,\boldsymbol{r}'',\boldsymbol{r}'')+(\boldsymbol{r}-\boldsymbol{R}_0,\boldsymbol{r}',\boldsymbol{r}''')=0.$

但前两项为 0，所以

$$(\boldsymbol{r}-\boldsymbol{R}_0,\boldsymbol{r}',\boldsymbol{r}''')=0.$$

由于上式与（ * ）式同时成立，所以 $\boldsymbol{r}',\boldsymbol{r}'',\boldsymbol{r}'''$ 共面，即

$$(\boldsymbol{r}',\boldsymbol{r}'',\boldsymbol{r}''')=0.$$

$$\tau=\frac{(\boldsymbol{r}',\boldsymbol{r}'',\boldsymbol{r}''')}{|\boldsymbol{r}'\times\boldsymbol{r}''|^2}=0,$$

故曲线（C）是平面曲线.

证法二　设曲线（C）：$\boldsymbol{r}=\boldsymbol{r}(s)$，依已知条件

$$(\boldsymbol{r}(s)-\boldsymbol{R}_0)\cdot\boldsymbol{\gamma}(s)=0, \qquad (**)$$

两边对 s 求微商：

$$\boldsymbol{\alpha}\cdot\boldsymbol{\gamma}+(\boldsymbol{r}-\boldsymbol{R}_0)\cdot(-\tau\boldsymbol{\beta})=0,$$

所以

$$\tau(\boldsymbol{r}-\boldsymbol{R}_0)\cdot\boldsymbol{\beta}=0.$$

（1）若 $\tau=0$，则（C）是平面曲线；

（2）若 $(\boldsymbol{r}-\boldsymbol{R}_0)\cdot\boldsymbol{\beta}=0$，两边对 s 求微商：

$$\boldsymbol{\alpha}\cdot\boldsymbol{\beta}+(\boldsymbol{r}-\boldsymbol{R}_0)\cdot(-k\boldsymbol{\alpha}+\tau\boldsymbol{\gamma})=0,$$

所以

$$(\boldsymbol{r}-\boldsymbol{R}_0)\cdot(-k\boldsymbol{\alpha})+(\boldsymbol{r}-\boldsymbol{R}_0)\cdot\tau\boldsymbol{\gamma}=0.$$

根据已知条件（ * * ）式，后一项为 0，所以

$$k(\boldsymbol{r}-\boldsymbol{R}_0)\cdot\boldsymbol{\alpha}=0.$$

但由所设 $(\boldsymbol{r}-\boldsymbol{R}_0)\perp\boldsymbol{\beta}$，$(\boldsymbol{r}-\boldsymbol{R}_0)\perp\boldsymbol{\gamma}$，所以

$$(\boldsymbol{r}-\boldsymbol{R}_0)\,/\!/\,\boldsymbol{\alpha},$$

故 $(\boldsymbol{r}-\boldsymbol{R}_0)\cdot\boldsymbol{\alpha}\neq 0$，从而 $k=0$，曲线（C）是直线.

11　例题中已给出解答.

12　证明：设曲线（C）：$\boldsymbol{r}=\boldsymbol{r}(s)$ 的曲率 $k=\text{cons } t\neq 0$，则其曲率中心的轨迹为

$$(C^*):\boldsymbol{r}^*=\boldsymbol{r}(s)+\frac{1}{k}\boldsymbol{\beta}(s).$$

上式两边对曲线（C^*）的自然参数 s^* 求微商，得

$$\dot{\boldsymbol{r}}^*=\left[\dot{\boldsymbol{r}}(s)+\frac{1}{k}\dot{\boldsymbol{\beta}}(s)\right]\frac{\mathrm{d}s}{\mathrm{d}s^*},$$

13　证明：因为$\boldsymbol{r}=\{1+3t+2t^2, 2-2t+5t^2, 1-t^2\}$，

$$\boldsymbol{r}'=\{3+4t, -2+10t, -2t\},$$
$$\boldsymbol{r}''=\{4, 10, -2\},$$
$$\boldsymbol{r}'''=\boldsymbol{0}.$$

从而 $\tau=0$，即曲线是平面曲线．令 $t=0$，则得

$$\boldsymbol{r}(0)=\{1,2,1\},$$
$$\boldsymbol{r}'(0)=\{3,-2,0\}.$$

作为平面曲线，它所在的平面即是它的密切平面，其方程为

$$\begin{vmatrix} x-1, & y-2 & z-1 \\ 3 & -2 & 0 \\ 4 & 10 & -2 \end{vmatrix}=0.$$

即
$$2x+3y+19z-27=0.$$

14　证法一　设曲线 $\Gamma: \boldsymbol{r}_1=\boldsymbol{r}_1(s_1), \Gamma_2: \boldsymbol{r}_2=\boldsymbol{r}_2(s_2)$．因为 $\boldsymbol{\alpha}_1 /\!/ \boldsymbol{\alpha}_2$，从而 $\boldsymbol{\alpha}_1=\pm\boldsymbol{\alpha}_2$，于是

$$\dot{\boldsymbol{\alpha}}_1=\pm\frac{\mathrm{d}\boldsymbol{\alpha}_2}{\mathrm{d}s_2}\cdot\frac{\mathrm{d}s_2}{\mathrm{d}s_1},$$

$$k_1\boldsymbol{\beta}_1=\pm k_2\boldsymbol{\beta}_2\frac{\mathrm{d}s_2}{\mathrm{d}s_1}.$$

于是有

$$\boldsymbol{\alpha}^*=\left[\boldsymbol{\alpha}+\frac{1}{k}(-k\boldsymbol{\alpha}+\tau\boldsymbol{\gamma})\right]\frac{\mathrm{d}s}{\mathrm{d}s^*}=\frac{1}{k}\tau\frac{\mathrm{d}s}{\mathrm{d}s^*}\boldsymbol{\gamma},$$

即　$\boldsymbol{\alpha}^* /\!/ \boldsymbol{\gamma}$，并且

$$\left|\frac{\mathrm{d}s}{\mathrm{d}s^*}\right|=\frac{k}{|\tau|}.$$

因为　$\boldsymbol{\alpha}^* /\!/ \boldsymbol{\gamma}$，从而　$\boldsymbol{\alpha}^*=\pm\boldsymbol{\gamma}$，上式两边对 s^* 求导，得

$$k^*\boldsymbol{\beta}^*=\pm\tau\boldsymbol{\beta}\frac{\mathrm{d}s}{\mathrm{d}s^*}.$$

所以

$$k^*=|\tau|\cdot\left|\frac{\mathrm{d}s}{\mathrm{d}s^*}\right|=|\tau|\cdot\frac{k}{|\tau|}=k.$$

因此，$\boldsymbol{\beta}_1 /\!/ \boldsymbol{\beta}_2$. 即 Γ_1，Γ_2 在对应点的主法线平行. 又 $\boldsymbol{\alpha}_1 /\!/ \boldsymbol{\alpha}_2$，所以 $\boldsymbol{\gamma}_1 /\!/ \boldsymbol{\gamma}_2$，即 Γ_1、Γ_2 在对应点处的副法线平行.

证法二　因为 $\boldsymbol{\alpha}_1 \times \boldsymbol{\alpha}_2 = \mathbf{0}$，所以

$$\dot{\boldsymbol{\alpha}}_1 \times \boldsymbol{\alpha}_2 + \boldsymbol{\alpha}_1 \times \dot{\boldsymbol{\alpha}}_2 \frac{\mathrm{d}s_2}{\mathrm{d}s_1} = \mathbf{0},$$

于是有

$$k_1 \boldsymbol{\beta}_1 \times \boldsymbol{\alpha}_2 + \boldsymbol{\alpha}_1 \times k_2 \boldsymbol{\beta}_2 \frac{\mathrm{d}s_2}{\mathrm{d}s_1} = \mathbf{0}.$$

从而 $\pm k_1 \boldsymbol{\beta}_1 \times \boldsymbol{\alpha}_1 \pm \boldsymbol{\alpha}_2 \times \boldsymbol{\beta}_2 \frac{\mathrm{d}s_2}{\mathrm{d}s_1} = \mathbf{0}$　（根据 $\boldsymbol{\alpha}_2 = \pm \boldsymbol{\alpha}_1$）.

$$\mp k_1 \boldsymbol{\gamma}_1 \pm \boldsymbol{\gamma}_2 \left(k_2 \frac{\mathrm{d}s_2}{\mathrm{d}s_1} \right) = \mathbf{0}.$$

因此 $\boldsymbol{\gamma}_1 /\!/ \boldsymbol{\gamma}_2$，又由于 $\boldsymbol{\alpha}_1 /\!/ \boldsymbol{\alpha}_2$，所以 $\boldsymbol{\beta}_1 /\!/ \boldsymbol{\beta}_2$.

15　证明：因为 $\boldsymbol{\beta}_1 /\!/ \boldsymbol{\beta}_2$，于是 $\boldsymbol{\alpha}_2 \perp \boldsymbol{\beta}_1$，$\boldsymbol{\alpha}_1 \perp \boldsymbol{\beta}_2$. 从而

$$\begin{aligned}\frac{\mathrm{d}(\boldsymbol{\alpha}_1 \cdot \boldsymbol{\alpha}_2)}{\mathrm{d}s_1} &= \dot{\boldsymbol{\alpha}}_1 \cdot \boldsymbol{\alpha}_2 + \boldsymbol{\alpha}_1 \cdot \left(\dot{\boldsymbol{\alpha}}_2 \frac{\mathrm{d}s_2}{\mathrm{d}s_1} \right) \\ &= k_1 \boldsymbol{\beta}_1 \cdot \boldsymbol{\alpha}_2 + \boldsymbol{\alpha}_1 \cdot k_2 \boldsymbol{\beta}_2 \frac{\mathrm{d}s_2}{\mathrm{d}s_1} \\ &= 0,\end{aligned}$$

所以 $\boldsymbol{\alpha}_1 \cdot \boldsymbol{\alpha}_2$ 为常数，即 $\boldsymbol{\alpha}_1$ 与 $\boldsymbol{\alpha}_2$ 作固定角.

16　证明：设曲线 $\Gamma: \boldsymbol{r} = \boldsymbol{r}(s)$，曲线 $\overline{\Gamma}: \boldsymbol{r}^* = \boldsymbol{r}^*(s^*)$. Γ 在 $\boldsymbol{r}(s)$ 的主法线与 $\overline{\Gamma}$ 在 $\boldsymbol{r}^*(s^*)$ 的副法线重合，则

$$\boldsymbol{r}^*(s^*) = \boldsymbol{r}(s) + \lambda(s)\boldsymbol{\beta}(s).$$

于是有

$$\dot{\boldsymbol{r}}^* \frac{\mathrm{d}s^*}{\mathrm{d}s} = \dot{\boldsymbol{r}} + \dot{\lambda}\boldsymbol{\beta} + \lambda\dot{\boldsymbol{\beta}},$$

$$\boldsymbol{\alpha}^* \frac{\mathrm{d}s^*}{\mathrm{d}s} = \boldsymbol{\alpha} + \dot{\lambda}\boldsymbol{\beta} + \lambda(-k\boldsymbol{\alpha} + \tau\boldsymbol{\gamma}).$$

因为 $\boldsymbol{\beta} /\!/ \boldsymbol{\gamma}^*$，于是 $\boldsymbol{\beta} \perp \boldsymbol{\alpha}^*$，$\boldsymbol{\beta} \perp \boldsymbol{\beta}^*$，上式两边点乘 $\boldsymbol{\beta}$，可得 $\dot{\lambda} = 0$，从而 λ 是常数. 设 $\lambda = \lambda_0$，则

$$\boldsymbol{\alpha}^* \frac{ds^*}{ds} = (1-\lambda_0 k)\boldsymbol{\alpha} + \lambda_0 \tau \boldsymbol{\gamma}.$$

上式两边对 s 求微商，可得

$$\dot{\boldsymbol{\alpha}}^* \left(\frac{ds^*}{ds}\right)^2 + \boldsymbol{\alpha}^* \frac{d^2 s^*}{ds^2} = (1-\lambda_0 k)^{\cdot} \boldsymbol{\alpha} + k(1-\lambda_0 k)\boldsymbol{\beta} + (\lambda_0 \tau)^{\cdot} \boldsymbol{\gamma} - \lambda_0 \tau^2 \boldsymbol{\beta}.$$

上式两边点乘 $\boldsymbol{\beta}$，可得

$$k(1-\lambda_0 k) - \lambda_0 \tau^2 = 0,$$

即

$$k = \lambda_0 (k^2 + \tau^2).$$

17 解 因为 $\boldsymbol{r} = \left\{a(t-\sin t), a(1-\cos t), 4a\cos\frac{t}{2}\right\}$,

$$\boldsymbol{r}' = \left\{a(1-\cos t), a\sin t, -2a\sin\frac{t}{2}\right\},$$

$$\boldsymbol{r}'' = \left\{a\sin t, a\cos t, -a\cos\frac{t}{2}\right\},$$

$$\boldsymbol{r}' \times \boldsymbol{r}'' = -2a^2\sin^2\frac{t}{2}\left\{\sin\frac{t}{2}, \cos\frac{t}{2}, 1\right\}.$$

$$k = \frac{1}{8a\left|\sin\frac{t}{2}\right|}.$$

当 $\frac{t}{2} = \frac{\pi}{2} + n\pi$，即 $t = \pi + 2n\pi = (2n+1)\pi$ 时，$\frac{1}{k} = \rho$ 最大.

18 解：因为 $\boldsymbol{r}(s)$ 在 s_0 点的泰勒展开式

$$\boldsymbol{r}(s_0+\Delta s) = \boldsymbol{r}(s_0) + \dot{\boldsymbol{r}}(s_0)\Delta s + \frac{1}{2!}\ddot{\boldsymbol{r}}(s_0)(\Delta s)^2 + \frac{1}{3!}[\dddot{\boldsymbol{r}}(s_0) + \boldsymbol{\varepsilon}(s_0, \Delta s)](\Delta s)^3,$$

于是 $\boldsymbol{r}(s_0+\Delta s) - \boldsymbol{r}(s_0)$

$$= \boldsymbol{\alpha}(s_0)\Delta s + \frac{1}{2}k_0\boldsymbol{\beta}(s_0)(\Delta s)^2 + \frac{1}{6}\{\dot{k}(s_0)\boldsymbol{\beta}(s_0) + k_0[-k_0\boldsymbol{\alpha}(s_0) + \tau_0\boldsymbol{\gamma}(s_0)] + [\varepsilon_1(s_0)\boldsymbol{\alpha}(s_0) + \varepsilon_2(s_0)\boldsymbol{\beta}(s_0) +$$

$$\varepsilon_3(s_0)\boldsymbol{\gamma}(s_0)]\}(\Delta s)^3$$

$$=\left[\Delta s-\frac{1}{6}k_0^2(\Delta s)^3+\frac{1}{6}\varepsilon_1(s_0)(\Delta s)^3\right]\boldsymbol{\alpha}(s_0)+$$

$$\left[\frac{1}{2}k_0(\Delta s)^2+\frac{1}{6}\dot{k}(s_0)(\Delta s)^3+\frac{1}{6}\varepsilon_2(s_0)(\Delta s)^3\right]\boldsymbol{\beta}(s_0)+$$

$$\left[\frac{1}{6}k_0\tau_0(\Delta s)^3+\frac{1}{6}\varepsilon_3(s_0)(\Delta s)^2\right]\boldsymbol{\gamma}(s_0).$$

设 z,x,y 分别是 $\boldsymbol{r}(s_0+\Delta s)$ 点到 $\boldsymbol{r}(s_0)$ 点的密切平面、法平面、从切平面的距离，则

$$x=|[\boldsymbol{r}(s_0+\Delta s)-\boldsymbol{r}(s_0)]\cdot\boldsymbol{\alpha}(s_0)|$$

$$=\left|\Delta s-\frac{1}{6}k_0^2(\Delta s)^3+\frac{1}{6}\varepsilon_1(s_0)(\Delta s)^3\right|,$$

$$y=|[\boldsymbol{r}(s_0+\Delta s)-\boldsymbol{r}(s_0)]\cdot\boldsymbol{\beta}(s_0)|$$

$$=\left|\frac{1}{2}k_0(\Delta s)^2+\frac{1}{6}\dot{k}(s_0)(\Delta s)^3+\frac{1}{6}\varepsilon_2(s_0)(\Delta s)^3\right|,$$

$$z=|[\boldsymbol{r}(s_0+\Delta s)-\boldsymbol{r}(s_0)]\cdot\boldsymbol{\gamma}(s_0)|$$

$$=\left|\frac{1}{6}k_0\tau_0(\Delta s)^3+\frac{1}{6}\varepsilon_3(s_0)(\Delta s)^3\right|.$$

当 $\Delta s\to0$ 时，$\boldsymbol{\varepsilon}(s_0)\to0$，即 $\varepsilon_1(s_0),\varepsilon_2(s_0),\varepsilon_3(s_0)\to0$ 所以，若 $k_0\neq0$，则以上三个距离的近似值分别为

$$x\approx|\Delta s|,$$

$$y\approx\left|\frac{1}{2}k_0(\Delta s)^2\right|=\frac{1}{2}k_0|\Delta s|^2,$$

$$z=\left|\frac{1}{6}k_0\tau_0(\Delta s)^3\right|=\frac{1}{6}k_0|\tau_0||\Delta s|^3.$$

若 $k_0=0,\dot{k}(s_0)\neq0$，则近似距离分别为

$$x\approx|\Delta s|,$$

$$y\approx\left|\frac{1}{6}\dot{k}(s_0)(\Delta s)^3\right|=\frac{1}{6}|\dot{k}(s_0)||\Delta s|^3,$$

$$z\approx\left|\frac{1}{6}k_0\tau_0(\Delta s)^3\right|=\frac{1}{6}k_0|\tau_0||\Delta s|^3.$$

习题 2.1

1 解 u-曲线为($v=v_0$)

$$r=\{u\cos v_0, u\sin v_0, bv_0\},$$

它是与 z 轴垂直相交的直线.

v-曲线($u=u_0$)为

$$r=\{u_0\cos v, u_0\sin v, bv\},$$

它是圆柱螺线.

2 证明 坐标曲线为

$$r=\{a(u+v_0), b(u-v_0), 2uv_0\},$$

$$r=\{a(u_0+v), b(u_0-v), 2u_0v\}.$$

它们都是直线族,又双曲抛物面上的直线必属于两族直母线之一,故曲面的坐标曲线就是它的直母线.

3 解 因为 $\boldsymbol{r}=\{a\cos\theta\cos\varphi, a\cos\theta\sin\varphi, a\sin\theta\}$,

$$r_\varphi=\{-a\cos\theta\sin\varphi, a\cos\theta\cos\varphi, 0\},$$

$$r_\theta=\{-a\sin\theta\cos\varphi, -a\sin\theta\sin\varphi, a\cos\theta\}.$$

所以,切平面的方程为

$$\begin{vmatrix} X-a\cos\theta\cos\varphi, & Y-a\cos\theta\sin\varphi, & Z-a\sin\theta \\ -a\cos\theta\sin\varphi & +a\cos\theta\cos\varphi & 0 \\ -a\sin\theta\cos\varphi & -a\sin\theta\sin\varphi & a\cos\theta \end{vmatrix}=0.$$

即 $$(\cos\theta\cos\varphi)X+(\cos\theta\sin\varphi)Y+\sin\theta Z-a=0.$$

法线方程为

$$\frac{X-a\cos\theta\cos\varphi}{\begin{vmatrix} a\cos\theta\cos\varphi & 0 \\ -a\sin\theta\sin\varphi & a\cos\theta \end{vmatrix}}=\frac{Y-a\cos\theta\sin\varphi}{\begin{vmatrix} 0 & -a\cos\theta & \sin\varphi \\ a\cos\theta & -a\sin\theta & \cos\varphi \end{vmatrix}}$$

$$=\frac{Z-a\sin\theta}{\begin{vmatrix} -a\cos\theta\sin\varphi & a\cos\theta\cos\varphi \\ -a\sin\theta\cos\varphi & -a\sin\theta\sin\varphi \end{vmatrix}},$$

习题 2.2

1 解 $\boldsymbol{r}=\{a(u+v),b(u-v),2uv\}$,

$\boldsymbol{r}_u=\{a,b,2v\}$,

$\boldsymbol{r}_v=\{a,-b,2u\}$.

$E=\boldsymbol{r}_u\cdot\boldsymbol{r}_u=a^2+b^2+4v^2$, $F=\boldsymbol{r}_u\cdot\boldsymbol{r}_v=a^2-b^2+4uv$,

$G=\boldsymbol{r}_v\cdot\boldsymbol{r}_v=a^2+b^2+4u^2$.

$\therefore\quad I=(a^2+b^2+4v^2)\mathrm{d}u^2+2(a^2-b^2+4uv)\mathrm{d}u\mathrm{d}v+(a^2+b^2+4u^2)\mathrm{d}v^2$.

2 解 $\boldsymbol{r}=\{u\cos v,u\sin v,bv\}$,

$\boldsymbol{r}_u=\{\cos v,\sin v,0\}$,

$\boldsymbol{r}_v=\{-u\sin v,u\cos v,b\}$.

$E=1$, $F=0$, $G=u^2+b^2$.

$\therefore\quad I=\mathrm{d}u^2+(u^2+b^2)\mathrm{d}v^2$.

又由于 $F=0$,所以坐标曲线互相垂直.

3 解 $\mathrm{I}=\mathrm{d}u^2+\sin\mathrm{h}^2u\mathrm{d}v^2$,沿曲面上曲线 $u=v$,有

$$\begin{aligned}\mathrm{d}s^2&=\mathrm{d}u^2+\sin\mathrm{h}^2u\mathrm{d}u^2\\&=(1+\sinh^2u)\mathrm{d}u^2\\&=\cosh^2u\mathrm{d}u^2.\end{aligned}$$

设曲线 $u=v$ 上两点 $A(u_1)$, $B(u_2)$ $(u_1<u_2)$,则曲线的弧长为

$$\begin{aligned}s&=\int_{u_1}^{u_2}\frac{\mathrm{d}s}{\mathrm{d}u}\mathrm{d}u=\int_{u_1}^{u_2}\sqrt{\cosh^2u}\,\mathrm{d}u\\&=\left|\sinh u\Big|_{u_1}^{u_2}\right|=|\sinh u_2-\sinh u_1|.\end{aligned}$$

4 解 由 $\mathrm{I} = \mathrm{d}u^2 + (u^2 + a^2)\mathrm{d}v^2$，得

$$E = 1, F = 0, G = u^2 + a^2.$$

曲线 $u + v = 0$，$u - v = 0$ 的交点为 $u = 0$，$v = 0$. 在交点(0,0)处，$E = 1$，$F = 0$，$G = a^2$.

由 $u + v = 0$ 得 $\frac{\mathrm{d}u}{\mathrm{d}v} = -1$，由 $u - v = 0$ 得 $\frac{\delta u}{\delta v} = 1$. 所以

$$\cos\varphi = \frac{E\frac{\mathrm{d}u}{\mathrm{d}v}\cdot\frac{\delta u}{\delta v} + F\left(\frac{\mathrm{d}u}{\mathrm{d}v} + \frac{\delta u}{\delta v}\right) + G}{\sqrt{E\left(\frac{\mathrm{d}u}{\mathrm{d}v}\right)^2 + 2F\frac{\mathrm{d}u}{\mathrm{d}v} + G}\sqrt{E\left(\frac{\delta u}{\delta v}\right)^2 + 2F\frac{\delta u}{\delta v} + G}}$$

$$= \frac{-1 + a^2}{1 + a^2}.$$

$$\theta = \arccos\frac{a^2 - 1}{a^2 + 1}.$$

5 解
$$r = \{x, y, axy\},$$
$$r_x = \{1, 0, ay\},$$
$$r_y = \{0, 1, ax\}.$$

在交点(x_0, y_0)处，

$$E = 1 + a^2 y_0^2, F = a^2 x_0 y_0, G = 1 + a^2 x_0^2.$$

$$\cos\theta = \frac{F}{\sqrt{EG}} = \frac{a^2 x_0 y_0}{\sqrt{(1 + a^2 y_0^2)(1 + a^2 x_0^2)}}.$$

$$\theta = \arccos\frac{a^2 x_0 y_0}{\sqrt{(1 + a^2 y_0^2)(1 + a^2 x_0^2)}}.$$

6 解 对于 u - 曲线：$\mathrm{d}v = 0$，$\mathrm{d}u \neq 0$. 代入 $E\mathrm{d}u\delta u + F(\mathrm{d}u\delta v + \mathrm{d}v\delta u) + G\mathrm{d}v\delta v = 0$中得

$$E\mathrm{d}u\delta u + F\mathrm{d}u\delta v = 0,$$

因为 $\mathrm{d}u \neq 0$，所以 u - 曲线的正交轨线的微分方程为：

$$E\delta u + F\delta v = 0.$$

同理可得 v - 曲线的正交轨线的微分方程为

$$F\delta u + G\delta v = 0.$$

7　证明　因为 du, dv 不同时为 0，不妨设 $dv \neq 0$. 由已知条件有

$$P\left(\frac{du}{dv}\right)^2 + 2Q\frac{du}{dv} + R = 0.$$

设上面关于 $\frac{du}{dv}$ 的二次方程的两根分别为 $\frac{du}{dv}, \frac{\delta u}{\delta v}$. 由韦达定理得

$$\frac{du}{dv} + \frac{\delta u}{\delta v} = -2\frac{Q}{P},$$

$$\frac{du}{dv} \cdot \frac{\delta u}{\delta v} = \frac{R}{P}.$$

代入　$E\frac{du}{dv}\frac{\delta u}{\delta v} + F\left(\frac{du}{dv} + \frac{\delta u}{\delta v}\right) + G = 0$ 中得

$$E\frac{R}{P} + F\left(-2\frac{Q}{P}\right) + G = 0,$$

即

$$ER - 2FQ + GP = 0.$$

8　证明　由于 $d\boldsymbol{r} = \boldsymbol{r}_u du + \boldsymbol{r}_v dv$，设 $d\boldsymbol{r}$ 是 $\boldsymbol{r}_u$、$\boldsymbol{r}_v$ 交角的平分线. 则

$$d\boldsymbol{r} \cdot \frac{\boldsymbol{r}_u}{\sqrt{E}} = d\boldsymbol{r}\frac{\boldsymbol{r}_v}{\sqrt{G}}.$$

所以

$$(\boldsymbol{r}_u du + \boldsymbol{r}_v dv) \cdot \frac{\boldsymbol{r}_u}{\sqrt{E}} = (\boldsymbol{r}_u du + \boldsymbol{r}_v dv) \cdot \frac{\boldsymbol{r}_v}{\sqrt{G}}.$$

$$\frac{Edu + Fdv}{\sqrt{E}} = \frac{Edu + Gdv}{\sqrt{G}}.$$

由此式得

$$\frac{E^2du^2 + 2EFdudv + F^2dv^2}{E} = \frac{F^2du^2 + 2FGdudv + G^2dv^2}{G}.$$

即

$$E(EG - F^2)du^2 = G(EG - F^2)dv^2,$$

由于 $EG - F^2 > 0$，故所求二等分角轨线的微分方程为

$$Edu^2 = Gdv^2.$$

10 解 先求球面的第一基本量.

$$r=\{R\cos\theta\cos\varphi,R\cos\theta\sin\varphi,R\sin\theta\},-\frac{\pi}{2}<\theta<\frac{\pi}{2},0<\varphi<2\pi.$$

$$r_\theta=\{-R\sin\theta\cos\varphi,-R\sin\theta\sin\varphi,R\cos\theta\},$$

$$r_\varphi=\{-R\cos\theta\sin\varphi,R\cos\theta\cos\varphi\ \ 0\},$$

$$E=R^2,F=0,G=R^2\cos^2\theta.$$

$$I=R^2\mathrm{d}\theta^2+R^2\cos^2\theta\mathrm{d}\varphi^2,$$

$$\sqrt{EG-F^2}=R^2\cos\theta.$$

$$\sigma=\iint_S\sqrt{EG-F^2}\mathrm{d}\varphi\mathrm{d}\theta=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\left[\int_0^{2\pi}R^2\cos\theta\mathrm{d}\varphi\right]\mathrm{d}\theta$$

$$=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}2\pi R^2\cos\theta\mathrm{d}\theta=4\pi R^2.$$

11 解 先计算两曲面的第一基本形式：

$$r=\{u\cos v,u\sin v,u+v\},$$

$$r_u=\{\cos v,\sin v,1\},$$

$$r_v=\{-u\sin v,u\cos v,1\},$$

$$E=2,F=1,G=1+u^2.$$

$$I=2\mathrm{d}u^2+2\mathrm{d}u\mathrm{d}v+(1+u^2)\mathrm{d}v^2.$$

$$r^*=\{t\cos\theta,t\sin\theta,\sqrt{t^2-1}\},$$

$$r_t^*=\left\{\cos\theta,\sin\theta,\frac{t}{\sqrt{t^2-1}}\right\},$$

$$r_\theta=\{-t\sin\theta,t\cos\theta,0\},$$

$$E^*=\frac{2t^2-1}{t^2-1},F^*=0,G^*=t^2.$$

$$I^*=\frac{2t^2-1}{t^2-1}\mathrm{d}t^2+t^2\mathrm{d}\theta^2.$$

将关系式

$$\theta = \arctan u + v, \quad d\theta = \frac{du}{1+u^2} + dv,$$

$$t = \sqrt{u^2+1}, \qquad dt = \frac{u}{\sqrt{u^2+1}}du,$$

代入 r^* 的第一基本形式 I^* 得

$$I^* = \frac{2(u^2+1)-1}{u^2+1-1}\cdot\left(\frac{u}{\sqrt{u^2+1}}\right)^2 du^2 + \left[\sqrt{u^2+1}\right]^2\cdot\left[\left(\frac{du}{1+u^2}\right)^2 + dv^2 + \frac{2}{1+u^2}du dv\right]$$

$$= 2du^2 + 2du dv + (1+u^2)dv^2.$$

由 $I = I^*$ 知两曲面等距等价.

习题 2.3

1 解 因为 $\boldsymbol{r} = \{\cosh u \cos v, \cosh u \sin v, v\}$,

$$\boldsymbol{r}_u = \{\sinh u \cos v, \sinh u \sin v, 1\},$$
$$\boldsymbol{r}_v = \{-\cosh u \sin v, \cosh u \cos v, 0\},$$
$$\boldsymbol{r}_{uu} = \{\cosh u \cos v, \cosh u \sin v, 0\},$$
$$\boldsymbol{r}_{uv} = \{-\sinh u \sin v, \sinh u \cos v, 0\},$$
$$\boldsymbol{r}_{vv} = \{-\cosh u \cos v, -\cosh u \sin v, 0\}.$$

所以有

$$E = \boldsymbol{r}_u\cdot\boldsymbol{r}_u = \cosh^2 u, F = 0, G = \boldsymbol{r}_v\cdot\boldsymbol{r}_v = \cosh^2 u.$$

因为

$$\boldsymbol{n} = \frac{\boldsymbol{r}_u\times\boldsymbol{r}_v}{|\boldsymbol{r}_u\times\boldsymbol{r}_v|} = \frac{1}{\cosh u}\{-\cos v, -\sin v, \sinh u\}.$$

所以

$$L = \boldsymbol{r}_{uu}\cdot\boldsymbol{n} = -1, M = \boldsymbol{r}_{uv}\cdot\boldsymbol{n} = 0, N = \boldsymbol{r}_{vv}\cdot\boldsymbol{n} = 1.$$

2 **解** 因为 $2x_3=5x_1^2+4x_1x_2+2x_2^2$,

$$x_3=\frac{5}{2}x_1^2+2x_1x_2+x_2^2,$$

$$p=\frac{\partial x_3}{\partial x_1}=5x_1+2x_2,$$

$$q=\frac{\partial x_3}{\partial x_2}=2x_1+2x_2,$$

$$r=\frac{\partial^2 x_3}{\partial x_1^2}=5,$$

$$s=\frac{\partial^2 x_3}{\partial x_1\partial x_2}=2,$$

$$t=\frac{\partial^2 x_3}{\partial x_2^2}=2.$$

在原点有

$$p=0,q=0,r=5,s=2,t=2.$$

所以

$E=1+p^2=1,F=pq=0,G=1+q^2=1,$

$L=\dfrac{r}{\sqrt{1+p^2+q^2}}=5,M=\dfrac{s}{\sqrt{1+p^2+q^2}}=2,N=\dfrac{t}{\sqrt{1+p^2+q^2}}=2,$

$\mathrm{I}=\mathrm{d}x_1^2+\mathrm{d}x_2^2,$

$\mathrm{II}=5\mathrm{d}x_1^2+4\mathrm{d}x_1\mathrm{d}x_2+2\mathrm{d}x_2^2.$

3 证明:由于 $\boldsymbol{r}=\{u\cos v,u\sin v,bv\}$,

$$\boldsymbol{r}_u=\{\cos v,\sin v,0\},$$

$$\boldsymbol{r}_v=\{-u\sin v,u\cos v,b\},$$

$$\boldsymbol{r}_{uu}=\{0,0,0\},$$

$$\boldsymbol{r}_{uv}=\{-\sin v,\cos v,0\},$$

$$\boldsymbol{r}_{vv}=\{-u\cos v,-u\sin v,0\}.$$

所以

$$E=1,F=0,G=u^2+b^2.$$

$$\boldsymbol{n}=\frac{1}{\sqrt{u^2+b^2}}\{b\sin v,-b\cos v,u\}.$$

$$L=0,M=-b,N=0.$$

故

$$EN-2FM+GL=0.$$

4 解 因为 $r=\{x,y,\frac{1}{2}(ax^2+by^2)\}$，所以

$$p=ax,q=by.$$
$$r=a,s=0,t=b.$$

在(0,0)点有

$$p_0=0,q_0=0,r_0=a,s_0=0,t_0=b,$$
$$E=1,F=0,G=1,L=a,M=0,N=b,$$
$$\mathrm{I}=\mathrm{d}x^2+\mathrm{d}y^2,$$
$$\mathrm{II}=a\mathrm{d}x^2+b\mathrm{d}y^2,$$

故在(0,0)点沿方向$(\mathrm{d}x:\mathrm{d}y)$的法曲率为：

$$k_n(\mathrm{d}x:\mathrm{d}y)=\frac{\mathrm{II}}{\mathrm{I}}=\frac{a\mathrm{d}x^2+b\mathrm{d}y^2}{\mathrm{d}x^2+\mathrm{d}y^2}=\frac{a\left(\frac{\mathrm{d}x}{\mathrm{d}y}\right)^2+b}{\left(\frac{\mathrm{d}x}{\mathrm{d}y}\right)^2+1}.$$

5 解 因为平面 π 与单位球面的交线为圆，其半径 $r=\sqrt{1-d^2}$，所以交线的曲率

$$k=\frac{1}{\sqrt{1-d^2}}.$$

因为球面 S 上任意点处沿任一切方向的法截线为 S 的大圆，所以 π 与 S 的交线上任一点处沿交线的切方向的法曲率 $k_n=1$.（取 $\boldsymbol{n}$ 指向球心）.

6 证明 对于球面 $\mathrm{I}=\{R\cos v\cos u,R\cos v\sin u,R\sin v\}$ 有

$$\mathrm{I}=R^2\cos^2 v\mathrm{d}u^2+R^2\mathrm{d}v^2,$$
$$\mathrm{II}=-(R\cos^2 v\mathrm{d}u^2+R\mathrm{d}v^2).$$

所以球面上任意点(u,v)沿任何方向$(\mathrm{d}u:\mathrm{d}v)$的法曲率为

$$k_n=\frac{\mathrm{II}}{\mathrm{I}}=-\frac{1}{R}.$$

又由于

$$k_n=\frac{\mathrm{II}}{\mathrm{I}}=\frac{L\mathrm{d}u^2+2M\mathrm{d}u\mathrm{d}v+N\mathrm{d}v^2}{E\mathrm{d}u^2+2F\mathrm{d}u\mathrm{d}v+G\mathrm{d}v^2}=-\frac{1}{R}.$$

化简得

$$(RL+E)\mathrm{d}u^2+2(RM+F)\mathrm{d}u\mathrm{d}v+(RN+G)\mathrm{d}v^2=0.$$

因为对任意 $\mathrm{d}u$、$\mathrm{d}v$ 都成立,故有

$$RL+E=RM+F=RN+G=0,$$

即

$$\frac{L}{E}=\frac{M}{F}=\frac{N}{G}=-\frac{1}{R}.$$

7 解 因为 $\boldsymbol{r}=\{u\cos v,u\sin v,bv\}$,

$$E=1,F=0,G=u^2+b^2,$$

$$L=0,M=\frac{-b}{\sqrt{u^2+b^2}},N=0.$$

由于 $L=N=0$,所以,正螺面的曲纹坐标网是渐近网,则一族渐近线是

$$\boldsymbol{r}=\{u_0\cos v,u_0\sin v,bv\},$$

这是螺旋线.另一族渐近线是

$$\boldsymbol{r}=\{u\cos v_0,u\sin v_0,bv_0\},$$

这是直线.

8 解 见3.3节例题.

9 证明:设空间曲线(C):$\boldsymbol{r}=\boldsymbol{r}(s)$.它的主法线曲面$\Sigma$为

$$\boldsymbol{r}=\boldsymbol{r}(s)+t\boldsymbol{\beta}(s),$$

$$\boldsymbol{r}_s=\boldsymbol{\alpha}(s)+t(-k\boldsymbol{\alpha}+\tau\boldsymbol{\gamma})$$

$$=(1-kt)\boldsymbol{\alpha}+t\tau\boldsymbol{\gamma},$$

$$\boldsymbol{r}_t=\boldsymbol{\beta}(s).$$

曲面的法向量

$$\begin{aligned}N &= r_t \times r_s = [(1-kt)\boldsymbol{\alpha} + t\tau\boldsymbol{\gamma}] \times \boldsymbol{\beta} \\ &= (1-kt)\boldsymbol{\gamma} - t\tau\boldsymbol{\alpha}.\end{aligned}$$

沿曲线(C)：$t=0$，$\boldsymbol{N}=\boldsymbol{\gamma}$. 曲线的主法向量与曲面法向量的夹角 $\theta = \angle(\boldsymbol{\beta},\boldsymbol{\gamma}) = \frac{\pi}{2}$. 由于

$$k_n = k\cos\theta = k\cos\frac{\pi}{2} = 0,$$

所以曲线(C)是曲面Σ的渐近曲线.

10　证明　因为 $z = f(x) + g(y)$，所以

$$p = f'(x), \qquad s = 0,$$

$$M = \frac{s}{\sqrt{1+p^2+q^2}} = 0.$$

曲面上的曲纹坐标网是共轭网. 故曲线族 $x=$常数，$y=$常数构成共轭网.

11　解　对于正螺面 $\boldsymbol{r} = \{u\cos v, u\sin v, cv\}$，

$$E=1, F=0, G=u^2+c^2.$$

$$L=0, M=\frac{-c}{\sqrt{u^2+c^2}}, N=0.$$

曲率线的方程为

$$\begin{vmatrix} dv^2 & -du\,dv & du^2 \\ 1 & 0 & u^2+c^2 \\ 0 & \frac{-c}{\sqrt{u^2+c^2}} & 0 \end{vmatrix} = 0,$$

化简得

$$-du^2 + (u^2+c^2)dv^2 = 0,$$

即

$$\frac{du}{\sqrt{u^2+c^2}} = \pm dv.$$

积分得

$$\ln|u+\sqrt{u^2+c^2}| = \pm v + c.$$

所求曲率线为

$$\ln|u+\sqrt{u^2+c^2}| + v = c_1,$$

$$\ln|u+\sqrt{u^2+c^2}| - v = c_2.$$

12 **解** 见3.5节例题.

13 **解** 因为 $\boldsymbol{r}=\left\{\frac{a}{2}(u-v),\frac{b}{2}(u+v),\frac{uv}{2}\right\}$,

$$\boldsymbol{r}_u=\left\{\frac{a}{2},\frac{b}{2},\frac{v}{2}\right\},$$

$$\boldsymbol{r}_v=\left\{-\frac{a}{2},\frac{b}{2},\frac{u}{2}\right\},$$

$$\boldsymbol{r}_{uu}=\{0,0,0\},$$

$$\boldsymbol{r}_{uv}=\left\{0,0,\frac{1}{2}\right\},$$

$$\boldsymbol{r}_{vv}=\{0,0,0\}.$$

所以 $E=\frac{1}{4}(a^2+b^2+v^2),F=\frac{1}{4}(-a^2+b^2+uv)$,

$$G=\frac{1}{4}(a^2+b^2+u^2),$$

$$L=0,M=\frac{ab}{\sqrt{EG-F^2}},N=0.$$

曲率线的微分方程为:

$$\begin{vmatrix} dv^2 & -du\,dv & du^2 \\ \frac{1}{4}(a^2+b^2+v^2) & \frac{1}{4}(-a^2+b^2+uv) & \frac{1}{4}(a^2+b^2+u^2) \\ 0 & \frac{ab}{\sqrt{EG-F^2}} & 0 \end{vmatrix}=0.$$

化简为

$$\frac{du}{\sqrt{a^2+b^2+u^2}}=\pm\frac{dv}{\sqrt{a^2+b^2+v^2}}.$$

积分得

$$\ln|u+\sqrt{a^2+b^2+u^2}|\pm\ln|v+\sqrt{a^2+b^2+v^2}|=C.$$

14　证明　设曲面上曲率线 Γ 的方程为

$$\boldsymbol{r}=\boldsymbol{r}(s),$$

因为$\angle(\boldsymbol{n},\boldsymbol{\gamma})=\theta$(定角),所以 $\boldsymbol{n}\cdot\boldsymbol{\gamma}=\cos\theta$.两边对 S 求微商得

$$\dot{\boldsymbol{n}}\cdot\boldsymbol{\gamma}+\boldsymbol{n}\cdot\dot{\boldsymbol{\gamma}}=0.$$

由罗德里格定理知 $\dot{\boldsymbol{n}}\cdot\boldsymbol{\gamma}=0$,故有

$$\dot{\boldsymbol{n}}\cdot(-\tau\boldsymbol{\beta})=0.$$

若 $\tau=0$,则曲线 Γ 为一平面曲线.

若 $\boldsymbol{n}\cdot\boldsymbol{\beta}=0$,则 $\boldsymbol{n}\perp\boldsymbol{\beta}$,即 Γ 是渐近线,又由已知 Γ 是曲率线,由于

$$\mathrm{d}\boldsymbol{n}=-k_N\mathrm{d}\boldsymbol{r},$$

所以 $\mathrm{d}\boldsymbol{n}=0$,$\boldsymbol{n}$ 为常向量.

$$\mathrm{d}(\boldsymbol{n}\cdot\boldsymbol{r})=\mathrm{d}\boldsymbol{n}\cdot\boldsymbol{r}+\boldsymbol{n}\cdot\mathrm{d}\boldsymbol{r}=0,$$

所以

$$\mathrm{d}(\boldsymbol{n}\cdot\boldsymbol{r})=0,\boldsymbol{n}\cdot\boldsymbol{r}=c.$$

即 Γ 为平面曲线.

15　证明　对于曲率线 $\Gamma:\boldsymbol{r}=\boldsymbol{r}(s)$,由于 $\boldsymbol{n}$ 垂直于切平面,$\boldsymbol{\gamma}$ 垂直于 Γ 的密切平面.依题意 $\boldsymbol{n}$ 与 $\boldsymbol{\gamma}$ 成定角.由上题知:Γ 是平面曲线.

16　解　见 3.6 节例题.

17　解　由抛物面 $z=a(x^2+y^2)$知

$$p=2ax,q=2ay,r=2a,s=0,t=2a.$$

在(0,0)点

$$p_0=0,q_0=0,r_0=2a,s_0=0,t_0=2a,$$

$$E=1,F=0,G=1,L=2a,M=0,N=2a.$$

代入主曲率公式,得

$$\begin{vmatrix}2a-k_N & 0\\ 0 & 2a-k_N\end{vmatrix}=0.$$

主曲率为 $k_1=2a,k_2=2a$.

18 证明 由欧拉公式知

$$k_n = k_1\cos^2\theta + k_2\sin^2\theta,$$

$$k_n^* = k_1\cos^2\left(\frac{\pi}{2}+\theta\right) + k_2\sin^2\left(\frac{\pi}{2}+\theta\right)$$

$$= k_1\sin^2\theta + k_2\cos^2\theta,$$

所以 $k_n + k_n^* = k_1 + k_2 = \text{const}$.

19 证明 因为 $k_n = k_1\cos^2\theta + k_2\sin^2\theta$,对于渐近曲线,$k_n = 0$,所以

$$k_1\cos^2\theta + k_2\sin^2\theta = 0,$$

$$\tan\theta = \pm\sqrt{-\frac{k_1}{k_2}},$$

渐近方向间的夹角 $\varphi = 2\theta$. 所以

$$\frac{k_1}{k_2} = -\tan^2\frac{\varphi}{2} = \text{const}$$

20 解 见 3.6 节例题.

21 解 因为 $z = axy$,所以

$$p = ay, q = ax, r = 0, s = a, t = 0.$$

在(0,0)点有

$$p_0 = 0, q_0 = 0, r_0 = 0, s_0 = a, t_0 = 0,$$

$$E = 1, F = 0, G = 1,$$

$$L = 0, M = a, N = 0.$$

曲面在(0,0)点的平均曲率 $H = 0$,高斯曲率 $K = -a^2$.

22 证明 因为 $H = \frac{1}{2}(k_1 + k_2) = 0$,所以

$$k_1 = -k_2.$$

当 $k_1 = -k_2 = 0$ 时为平点.

当 $k_1 = -k_2 \neq 0$ 时,$K = k_1 \cdot k_2 = -k_1^2 < 0$,为双曲点.

23 证法一 由于

$$H=\frac{LG-2MF+NE}{2(EG-F^2)}=0,$$

所以

$$LG-2MF+NE=0,$$

$$E\left(\frac{N}{L}\right)+F\left(-\frac{2M}{L}\right)+G=0. \tag{1}$$

对于渐近线

$$L\mathrm{d}u^2+2M\mathrm{d}u\mathrm{d}v+N\mathrm{d}v^2=0,$$

有

$$\left(\frac{\mathrm{d}u}{\mathrm{d}v}\right)^2+\frac{2M}{L}\left(\frac{\mathrm{d}u}{\mathrm{d}v}\right)+\frac{N}{L}=0.$$

于是有

$$\begin{cases}\dfrac{\mathrm{d}u}{\mathrm{d}v}+\dfrac{\delta u}{\delta v}=-\dfrac{2M}{L},\\[2mm]\dfrac{\mathrm{d}u}{\mathrm{d}v}\cdot\dfrac{\delta u}{\delta v}=\dfrac{N}{L},\end{cases} \tag{2}$$

将(2)式代入(1)式,得

$$E\mathrm{d}u\delta u+F(\mathrm{d}u\delta v+\mathrm{d}v\delta u)+G\mathrm{d}v\delta v=0.$$

此式说明两个方向$(\mathrm{d}u:\mathrm{d}v)$,$(\delta u:\delta v)$互相垂直,故曲面上的渐近网构成正交网.

证法二 由

$$H=\frac{LG-2MF+NE}{2(EG-F^2)}=0,$$

得

$$LG-2MF+NE=0. \tag{1}$$

选取坐标网为渐近网,则 $L=N=0$. 由(1)式可得

$$-MF=0.$$

因为 $M\neq0$(非脐点),所以 $F=0$. 坐标网为正交网,即渐近网是正交网.

证法三 由

$$H=\frac{LG-2MF+NE}{2(EG-F^2)}=\frac{k_1+k_2}{2}=0,$$

得

$$k_1=-k_2,$$

$$\tan\theta=\pm\sqrt{-\frac{k_1}{k_2}}=\pm1.$$

所以

$$\theta=\pm\frac{\pi}{4},$$

$$2\theta = \pm \frac{\pi}{2}.$$

即渐近线的夹角为$\frac{\pi}{2}$,渐近网为正交网.

24 解 因为

$$\boldsymbol{r} = \{(b + a\cos\varphi)\cos\theta, (b + a\cos\varphi)\sin\theta, a\sin\varphi\},$$
$$\boldsymbol{r}_\varphi = \{-a\sin\varphi\cos\theta, -a\sin\varphi\sin\theta, a\cos\varphi\},$$
$$\boldsymbol{r}_\theta = \{-(b + a\cos\varphi)\sin\theta, (b + a\cos\varphi)\cos\theta, 0\},$$
$$\boldsymbol{r}_{\varphi\varphi} = \{-a\cos\varphi\cos\theta, -a\cos\varphi\sin\theta, -a\sin\varphi\},$$
$$\boldsymbol{r}_{\varphi\theta} = \{a\sin\varphi\sin\theta, -a\sin\varphi\cos\theta, 0\},$$
$$\boldsymbol{r}_{\theta\theta} = \{-(b + a\cos\varphi)\cos\theta, -(b + a\cos\varphi)\sin\theta, 0\}.$$

则 $E = a^2, F = 0, G = (b + a\cos\varphi)^2,$

$L = a, M = 0, N = (b + a\cos\varphi)\cos\varphi.$

因为$b > a > 0$,又$-1 \leqslant \cos\varphi < 1$.所以

$$a(b + a\cos\varphi) > 0.$$

高斯曲率

$$K = \frac{LN - M^2}{EG - F^2} = \frac{a(b + a\cos\varphi)\cos\varphi}{a^2(b + a\cos\varphi)^2} = \frac{\cos\varphi}{a(b + a\cos\varphi)}.$$

当 $\left.\begin{array}{l} 0 \leqslant \varphi < \frac{\pi}{2} \\ \frac{3\pi}{2} < \varphi < 2\pi \end{array}\right\}$时,$K > 0$,是椭圆点;

当 $\frac{\pi}{2} < \varphi < \frac{3}{2}\pi$ 时,$K < 0$,是双曲点;

当 $\varphi = \frac{\pi}{2}$或 $\varphi = \frac{3}{2}\pi$ 时,$K = 0$,是抛物点.

25 证明 因为 $\boldsymbol{r}=\{g(t)\cos\theta, g(t)\sin\theta, f(t)\}$.

$\boldsymbol{r}_t=\{g'(t)\cos\theta, g'(t)\sin\theta, f'(t)\}$,

$\boldsymbol{r}_\theta=\{-g(t)\sin\theta, g(t)\cos\theta, 0\}$.

$E=[g'(t)]^2+[f'(t)]^2, F=0, G=[g(t)]^2$,

$I=([g'(t)]^2+[f'(t)]^2)\mathrm{d}t^2+[g(t)]^2\mathrm{d}\theta^2$.

将上式改写成

$$I=g^2(t)\left(\frac{g'^2+f'^2}{g^2}\mathrm{d}t^2+\mathrm{d}\theta^2\right).$$

作参数变换

$$u=\int\frac{\sqrt{g'^2+f'^2}}{g}\mathrm{d}t, v=\theta.$$

则

$$\mathrm{d}u=\frac{\sqrt{g'^2+f'^2}}{g}\mathrm{d}t, v=\theta$$

第一基本形式变为

$$\mathrm{I}=g^2(\mathrm{d}u^2+\mathrm{d}v^2).$$

26 证明 设 $\boldsymbol{n}_1$、$\boldsymbol{n}_2$分别是曲面 S_1、S_2的单位法向量. 设 $\boldsymbol{n}_1\cdot\boldsymbol{n}_2=\cos\varphi$. S_1与 S_2的交线(C)是 S_1的一条曲率线.

先证明充分性,即证明当 $\varphi=\varphi_0=\mathrm{const}$,则$(C)$是 S_2的曲率线.

由 $\frac{\mathrm{d}}{\mathrm{d}s}(\boldsymbol{n}_1\cdot\boldsymbol{n}_2)=\frac{\mathrm{d}\boldsymbol{n}_1}{\mathrm{d}s}\cdot\boldsymbol{n}_2+\boldsymbol{n}_1\cdot\frac{\mathrm{d}\boldsymbol{n}_2}{\mathrm{d}s}$,根据罗德理格定理

$$\frac{\mathrm{d}\boldsymbol{n}_1}{\mathrm{d}s}=-k_n\frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}s},$$

有

$$-k_n\frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}s}\cdot\boldsymbol{n}_2+\boldsymbol{n}_1\cdot\frac{\mathrm{d}\boldsymbol{n}_2}{\mathrm{d}s}=0,$$

但是 $\frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}s}=\boldsymbol{\alpha}$,且垂直于 $\boldsymbol{n}_2$于是有 $\boldsymbol{n}_1\cdot\frac{\mathrm{d}\boldsymbol{n}_2}{\mathrm{d}s}=0$.

即 $\boldsymbol{n}_1\perp\frac{\mathrm{d}\boldsymbol{n}_2}{\mathrm{d}s}$,又由于$\frac{\mathrm{d}\boldsymbol{n}_2}{\mathrm{d}s}\perp\boldsymbol{n}_2$,所以

$$\frac{\mathrm{d}\boldsymbol{n}_2}{\mathrm{d}s}\,/\!/\,\frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}s},$$

即 $$\frac{\mathrm{d}\boldsymbol{n}_2}{\mathrm{d}s}=\lambda\frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}s}.$$
由罗德里格定理知(C)也是 S_2的曲率线.

下面证明必要性.

由

$$\frac{\mathrm{d}}{\mathrm{d}s}(\boldsymbol{n}_1\cdot\boldsymbol{n}_2)=\frac{\mathrm{d}\boldsymbol{n}_1}{\mathrm{d}s}\cdot\boldsymbol{n}_2+\boldsymbol{n}_1\cdot\frac{\mathrm{d}\boldsymbol{n}_2}{\mathrm{d}s}.$$
因(C)是 S_1的曲率线,又是(S_2)的曲率线,所以

$$\frac{\mathrm{d}\boldsymbol{n}_1}{\mathrm{d}s}=\lambda_1\frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}s},\quad \frac{\mathrm{d}\boldsymbol{n}_2}{\mathrm{d}s}=\lambda_2\frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}s},$$
又$\frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}s}\perp\boldsymbol{n}_1,\frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}s}\perp\boldsymbol{n}_2$,所以

$$\frac{\mathrm{d}}{\mathrm{d}s}(\boldsymbol{n}_1\cdot\boldsymbol{n}_2)=0,\boldsymbol{n}_1\cdot\boldsymbol{n}_2=\cos\varphi=\mathrm{const}.$$

27 解 根据曲面上渐近线的性质可知:沿曲面的渐近线的密切平面与曲面的切平面重合,于是有

$$\boldsymbol{n}=\pm\boldsymbol{\gamma}.$$
两边对 s 求微商,

$$\dot{\boldsymbol{n}}=\pm\dot{\boldsymbol{\gamma}}=\pm(-\tau\boldsymbol{\beta})=\mp(\tau\boldsymbol{\beta}).$$
所以

$$\dot{\boldsymbol{n}}\cdot\dot{\boldsymbol{n}}=\tau^2.$$
即沿渐近线 $$\mathrm{III}=\mathrm{d}\boldsymbol{n}^2=\tau^2,$$
$$\mathrm{I}=\mathrm{d}\boldsymbol{r}^2=\boldsymbol{\alpha}^2=1,$$
$$\mathrm{II}=0.$$
代入公式 $$\mathrm{III}-2H\,\mathrm{II}+K\,\mathrm{I}=0,$$
得 $$\tau^2+K=0.$$
即 $$\tau=\pm\sqrt{-K}.$$

28 证明 对于曲面的球面表示 $\boldsymbol{n}(u,v)$,有

$$\boldsymbol{n}_u\times\boldsymbol{n}_v=K(\boldsymbol{r}_u\times\boldsymbol{r}_v).$$
对于简单曲面,$\boldsymbol{r}_u\times\boldsymbol{r}_v\neq0$.对于曲面上的非抛物点,$K\neq0$.所以

$$\boldsymbol{n}_u\times\boldsymbol{n}_v\neq0.$$
所以曲面上的点与其球面象上的点是一一对应的.

习题 2.4

1　证明:因为 $\boldsymbol{r}=\left\{u^2+\frac{1}{3}v,2u^3+uv,u^4+\frac{2}{3}u^2v\right\}$可以改写成

$$\boldsymbol{r}=\{u^2,2u^3,u^4\}+v\left\{\frac{1}{3},u,\frac{2}{3}u^2\right\}$$
$$=\boldsymbol{a}(u)+v\boldsymbol{b}(u).$$

所以

$$\boldsymbol{a}'(u)=\{2u,6u^2,4u^3\},$$
$$\boldsymbol{b}'(u)=\left\{0,1,\frac{4}{3}u\right\},$$
$$(\boldsymbol{b}',\boldsymbol{a}',\boldsymbol{b})=0,$$

所以曲面是可展曲面.

2　证明　因为 $\boldsymbol{r}=\{\cos v-(u+v)\sin v,\sin v+(u+v)\cdot\cos v,u+2v\}$可以改写成

$$\boldsymbol{r}=\{\cos v-v\sin v,\sin v+v\cos v,2v\}+u\{-\sin v,\cos v,1\}$$
$$=\boldsymbol{a}(v)+u\boldsymbol{b}(v).$$

则　$\boldsymbol{a}'(v)=\{-2\sin v-v\cos v,2\cos v-v\cos v,2\}$,

$\boldsymbol{b}'(v)=\{-\cos v,-\sin v,0\}$,

$(\boldsymbol{a}',\boldsymbol{b}',\boldsymbol{b})=0$,

所以此曲面是可展曲面.

3　证明:因为 $\boldsymbol{r}=\{u\cos v,u\sin v,av+b\}$可以改写成

$$\boldsymbol{r}=\{0,0,av+b\}+u\{\cos v,\sin v,0\}$$
$$=\boldsymbol{a}(v)+u\boldsymbol{b}(v).$$

则

$$\boldsymbol{a}'(v)=\{0,0,a\},$$
$$\boldsymbol{b}'(v)=\{-\sin v,\cos v,0\},$$
$$(\boldsymbol{a}',\boldsymbol{b},\boldsymbol{b}')=a\neq0,$$

故曲面不可展.

4　证明　设有空间挠曲线 $\boldsymbol{a}=\boldsymbol{a}(s)$,由它生成的主法线曲

面的方程为

$$\boldsymbol{r}=\boldsymbol{a}(s)+v\boldsymbol{\beta}(s),$$

$$\dot{\boldsymbol{a}}(s)=\boldsymbol{\alpha}(s),\dot{\boldsymbol{\beta}}(s)=-k\boldsymbol{\alpha}+\tau\boldsymbol{\gamma},$$

$$(\boldsymbol{a}',\boldsymbol{b},\boldsymbol{b}')=(\boldsymbol{\alpha},\boldsymbol{\beta},-k\boldsymbol{\alpha}+\tau\boldsymbol{\gamma})=\tau\neq 0.$$

故曲面不可展.

曲线的副法线曲面为

$$\boldsymbol{r}=\boldsymbol{a}(s)+v\boldsymbol{\gamma}(s),$$

$$\dot{\boldsymbol{\gamma}}(s)=-\tau\boldsymbol{\beta},$$

$$(\boldsymbol{a}',\boldsymbol{b},\boldsymbol{b}')=(\boldsymbol{\alpha},\boldsymbol{\gamma},-\tau\boldsymbol{\beta})=\tau\neq 0.$$

故曲面不可展.

5 解 平面族 $x\cos\alpha+y\sin\alpha-z\sin\alpha=1$.

$$\begin{cases}F(x,y,z,\alpha)=x\cos\alpha+y\sin\alpha-z\sin\alpha-1=0,\\ F'_\alpha=-x\sin\alpha+y\cos\alpha-z\cos\alpha=0,\end{cases}$$

即

$$\begin{cases}x\cos\alpha+(y-z)\sin\alpha-1=0,\\ -x\sin\alpha+(y-z)\cos\alpha=0,\end{cases}$$

解得 $$x^2+(y-z)^2=1.$$

曲面上的点都是正常点,故上式便是所求包络的方程.

6 解 平面族 $a^2x+2ay+2z=2a$.

$$\begin{cases}F(x,y,z,a)=a^2x+2ay+2z-2a=0,\\ F_a(x,y,z,a)=2ax+2y-2=0,\end{cases}$$

即

$$\begin{cases}a^2x+2ay+2z-2a=0,\\ a=\dfrac{1-y}{x}.\end{cases}$$

解得 $$2xz-y^2+2y-1=0.$$

所求包络是锥面 $$2xz-(y-1)^2=0.$$

7 证明 设柱面的方程为

$$\boldsymbol{r}=\boldsymbol{a}(u)+v\boldsymbol{b}(u).$$

其中 $\boldsymbol{b}(u)$为常向量. $\boldsymbol{b}'(u)=0$. 所以

$$(\boldsymbol{a}',\boldsymbol{b},\boldsymbol{b}')=0,$$

柱面是可展曲面.

设锥面的方程为

$$\boldsymbol{r}=\boldsymbol{a}(u)+v\boldsymbol{b}(u),$$

其中 $\boldsymbol{a}(u)$为常向量, $\boldsymbol{a}'(u)=0$. 故

$$(\boldsymbol{a}',\boldsymbol{b},\boldsymbol{b}')=0,$$

锥面是可展面.

设曲线的切线曲面为

$$\boldsymbol{r}=\boldsymbol{a}(u)+v\boldsymbol{b}(u).$$

其中 $\boldsymbol{a}'/\!/\boldsymbol{b}$, $\boldsymbol{a}'=\lambda\boldsymbol{b}$, 故$(\boldsymbol{a}',\boldsymbol{b},\boldsymbol{b}')=0$, 所以曲面是可展曲面.

8　证明: 因为 $\boldsymbol{r}_{uu}=\boldsymbol{r}_{uv}=0$, 所以曲面的第二基本量 $L=M=0$. 可见 $K=0$. 故曲面为可展面, 它的方程可写成

$$\boldsymbol{r}=\boldsymbol{a}(u)+v\boldsymbol{b}(u),$$

$$\boldsymbol{r}_u=\boldsymbol{a}'+v\boldsymbol{b}',\qquad \boldsymbol{r}_{uu}=\boldsymbol{a}''+v\boldsymbol{b}''=0,$$

$$\boldsymbol{r}_{uv}=\boldsymbol{b}'=0,$$

则 $\boldsymbol{b}'=0$, $\boldsymbol{b}=$常向量, 所以曲面是柱面.

习题 2.5

1　解　由于 $\mathrm{d}S^2=\mathrm{d}\rho^2+\rho^2\mathrm{d}\theta^2$, 即 $E(\rho,\theta)=1$, $F(\rho,\theta)=0$, $G(\rho,\theta)=\rho^2$, 所以

$$\Gamma_{11}^1=0,\Gamma_{11}^2=0,\Gamma_{12}^1=0,$$

$$\Gamma_{12}^2=\frac{1}{\rho},\Gamma_{22}^1=\rho,\Gamma_{22}^2=0.$$

2　证明　由于

$$\mu_i^j=-\sum_k g^{jk}L_{ik},$$

则有

$$\mu_1^1=\frac{-LG+MF}{EG-F^2},\quad \mu_1^2=\frac{LF-ME}{EG-F^2},$$

$$\mu_2^1=\frac{NF-MG}{EG-F^2},\quad \mu_2^2=\frac{-NE+MF}{EG-F^2}.$$

$$\det(\mu_i^j)=\frac{(-LG+MF)(-NE+MF)}{(EG-F^2)^2}-\frac{(LF-ME)(NF-MG)}{(EG-F^2)^2}$$

$$=\frac{LN-M^2}{EG-F^2}=K.$$

3 证明 由于

$$\mu_1^1=\frac{-LG+MF}{EG-F^2},\quad \mu_2^2=\frac{-NE+MF}{EG-F^2},$$

所以 $\frac{1}{2}(\mu_1^1+\mu_2^2)=\frac{1}{2}\frac{-LG+MF-NE+MF}{EG-F^2}$

$$=\frac{1}{2}\left(\frac{2MF-LG-NE}{EG-F^2}\right)=-H.$$

4 证明 (1) 由第一类黎曼曲率张量的定义

$$R_{ijk}^l=\frac{\partial\Gamma_{ij}^l}{\partial u^k}-\frac{\partial\Gamma_{ik}^l}{\partial u^j}+\sum_p(\Gamma_{ij}^p\Gamma_{pk}^l-\Gamma_{ik}^p\Gamma_{pj}^l),$$

有 $R_{mijk}=\sum_l g_{ml}R_{ijk}^l$

$$=\sum_l g_{ml}\left[\frac{\partial\Gamma_{ij}^l}{\partial u^k}-\frac{\partial\Gamma_{ik}^l}{\partial u^j}+\sum_p(\Gamma_{ij}^p\Gamma_{pk}^l-\Gamma_{ik}^p\Gamma_{pj}^l)\right].\tag{1}$$

由于 $[ij,m]=\sum_l\Gamma_{ij}^l g_{lm}$,

所以

$$\frac{\partial[ij,m]}{\partial u^k}=\sum_l\frac{\partial\Gamma_{ij}^l}{\partial u^k}g_{ml}+\sum_l\frac{\partial g_{lm}}{\partial u^k}\Gamma_{ij}^l.$$

即

$$\sum_l g_{ml}\frac{\partial\Gamma_{ij}^l}{\partial u^k}=\frac{\partial[ij,m]}{\partial u^k}-\sum\frac{\partial g_{lm}}{\partial u^k}\Gamma_{ij}^l.$$

代入(1)式得

$$R_{mijk}=\frac{\partial[ij,m]}{\partial u^k}-\frac{\partial[ik,m]}{\partial u^j}-\sum\frac{\partial g_{lm}}{\partial u^k}\Gamma_{ij}^l+$$

$$\sum_l\frac{\partial g_{ml}}{\partial u^j}\Gamma_{ik}^l+\sum_p[pk,m]\Gamma_{ij}^p-$$

$$\sum_p[p_j,m]\Gamma_{ik}^p.\tag{2}$$

由于

$$[ij,m]=\frac{1}{2}\left(\frac{\partial g_{im}}{\partial u^j}+\frac{\partial g_{jm}}{\partial u^i}-\frac{\partial g_{ij}}{\partial u^m}\right),$$

$$[ik,m]=\frac{1}{2}\left(\frac{\partial g_{im}}{\partial u^k}+\frac{\partial g_{km}}{\partial u^i}-\frac{\partial g_{ik}}{\partial u^m}\right),$$

$$\frac{\partial g_{ml}}{\partial u^k}=[ik,m]+[mk,l],$$

$$\frac{\partial g_{ml}}{\partial u^j}=[lj,m]+[mj,l],$$

代入(2)式得

$$R_{mijk}=\frac{1}{2}\frac{\partial}{\partial k}\left(\frac{\partial g_{im}}{\partial u^j}+\frac{\partial g_{jm}}{\partial u^i}-\frac{\partial g_{ij}}{\partial u^m}\right)-\frac{1}{2}\frac{\partial}{\partial u^j}\left(\frac{\partial g_{im}}{\partial u^k}+\right.$$

$$\left.\frac{\partial g_{km}}{\partial u^i}-\frac{\partial g_{ik}}{\partial u^m}\right)-([lk,m]+[mk,l])\Gamma_{ij}^l+$$

$$([lj,m]+[mj,l])\Gamma_{ik}^l+[pk,m]\Gamma_{ij}^p-[p_j,m]\Gamma_{ik}^p,$$

$$=\frac{1}{2}\left(\frac{\partial^2 g_{jm}}{\partial u^i\partial u^k}+\frac{\partial^2 g_{ik}}{\partial u^j\partial u^k}-\frac{\partial^2 g_{ij}}{\partial u^m\partial u^k}-\frac{\partial^2 g_{km}}{\partial u^i\partial u^j}\right)+$$

$$\sum_l[l_j,m]\Gamma_{ik}^l+\sum_l[mj,l]\Gamma_{ik}^l+\sum_p[pk,m]\Gamma_{ij}^p-$$

$$\sum_p[p_j,m]\Gamma_{ik}^p-\sum_l[lk,m]\Gamma_{ij}^l-\sum_l[mk,l]\Gamma_{ij}^l$$

$$=\frac{1}{2}\left(\frac{\partial^2 g_{jm}}{\partial u^i\partial u^k}+\frac{\partial^2 g_{im}}{\partial u^j\partial u^k}-\frac{\partial^2 g_{ij}}{\partial u^m\partial u^k}-\frac{\partial^2 g_{km}}{\partial u^i\partial u^j}\right)+$$

$$\sum[mj,p]\Gamma_{ik}^p-\sum[mk,p]\Gamma_{ij}^p.$$

5 证明 对于 $\mathbf{R}^3$ 中的曲面来说，R_{mijk} 中的本质分量只有一个，即

$$R_{1212}=-K_g.(g=g_{11}g_{22}-g_{21}g_{21}).$$

因此可以一般地表示为

$$R_{mijk}=-K(g_{mj}g_{ik}-g_{mk}g_{ij}).$$

但是

$$\begin{aligned}R_{ijk}^l&=g^{ml}R_{mijk}\\&=g^{ml}[-K(g_{mj}g_{ik}-g_{mk}g_{ij})]\\&=-K(g^{ml}g_{mj}g_{ik}-g^{ml}g_{mk}g_{ij})\end{aligned}$$

$$= -K(\delta_j^l g_{ik} - \delta_k^l g_{ij}).$$

所以

$$R_{ijk}^l = -K(\delta_j^l g_{ik} - \delta_k^l g_{ij}).$$

6 （1）证明　由高斯公式

$$\frac{\partial \Gamma_{ij}^l}{\partial u^k} - \frac{\partial \Gamma_{ik}^p}{\partial u^j} + \sum_p (\Gamma_{ij}^p \Gamma_{pk}^l - \Gamma_{ik}^p - \Gamma_{pj}^l)$$

$$= \sum_m g^{ml}[L_{ij}L_{mk} - L_{ik}L_{mj}].$$

取 $k=2, j=1, l=2, i=1$ 则

$$(\Gamma_{11}^2)_v - (\Gamma_{12}^2)_u + \Gamma_{11}^1\Gamma_{12}^2 + \Gamma_{11}^2\Gamma_{22}^2 - \Gamma_{12}^1\Gamma_{11}^2 - (\Gamma_{12}^2)^2$$

$$= g^{m2}[L_{11}L_{2m} - L_{12}L_{1m}]$$

$$= g^{12}[L_{11}L_{21} - L_{12}L_{11}] + g^{22}[L_{11}L_{22} - L_{12}L_{12}]$$

$$= g^{22}[LN - M^2]$$

$$= \frac{1}{g}g_{11}[LN - M^2]$$

$$= E\frac{LN - M^2}{EG - F^2} = EK.$$

（2）证明　由于

$$\frac{1}{\sqrt{EG-F^2}}\left[\frac{\partial}{\partial v}\left(\frac{\sqrt{EG-F^2}}{E}\Gamma_{11}^2\right) - \frac{\partial}{\partial u}\left(\frac{\sqrt{EG-F^2}}{E}\Gamma_{12}^2\right)\right]$$

$$= \frac{1}{\sqrt{g}}\left[\frac{\partial}{\partial u^2}\left(\frac{\sqrt{g}}{g_{11}}\Gamma_{11}^2\right) - \frac{\partial}{\partial u^1}\left(\frac{\sqrt{g}}{g_{11}}\Gamma_{12}^2\right)\right]$$

$$= \frac{1}{\sqrt{g}}\left[\frac{\frac{\partial}{\partial u^2}g_{11}\cdot\sqrt{g} - \frac{\partial}{\partial u^2}\sqrt{g}\cdot g_{11}}{(g_{11})^2}\Gamma_{11}^2 + \frac{\sqrt{g}}{g_{11}}\cdot\frac{\partial \Gamma_{11}^2}{\partial u^2} - \right.$$

$$\left.\frac{\frac{\partial}{\partial u^1}g_{11}\sqrt{g} - g_{11}\frac{\partial}{\partial u^1}\sqrt{g}}{(g_{11})^2}\Gamma_{11}^2 - \frac{\sqrt{g}}{g_{11}}\cdot\frac{\partial \Gamma_{12}^2}{\partial u^1}\right]$$

$$= \frac{1}{\sqrt{g}}\left[\left(\frac{\sqrt{g}\frac{\partial}{\partial u^2}g_{11}}{(g_{11})^2}\Gamma_{11}^2 - \frac{\sqrt{g}\frac{\partial}{\partial u^1}g_{11}}{(g_{11})^2}\Gamma_{12}^2\right) + \right.$$

$$\frac{1}{g_{11}}\left(\frac{\partial}{\partial u^1}\sqrt{g}\Gamma_{12}^2-\frac{\partial}{\partial u^1}\sqrt{g}\Gamma_{11}^2\right)+\frac{\sqrt{g}}{g_{11}}\left(\frac{\partial}{\partial u^2}\Gamma_{11}^2-\frac{\partial}{\partial u^1}\Gamma_{12}^2\right)\Bigg].\quad(1)$$

由于$\frac{\partial}{\partial u^1}\sqrt{g}=\frac{\partial}{\partial u^1}\sqrt{EG-F^2}$

$$=\frac{1}{2\sqrt{EG-F^2}}[E_uG+EG_u-2FF_u]$$

$$=\frac{1}{2\sqrt{EG-F^2}}[GE_u-2FF_u+FF_v+EG_v-FF_v]$$

$$=(\Gamma_{11}^1+\Gamma_{12}^2)\sqrt{g}.\quad(2)$$

同理 $\frac{\partial}{\partial u^2}\sqrt{g}=\frac{\partial}{\partial u^2}\sqrt{EG-F^2}$

$$=\frac{1}{2\sqrt{EG-F^2}}(E_vG+EG_v-2FF_v)$$

$$=\frac{1}{2\sqrt{EG-F^2}}(GE_v-FG_u+EG_v-2FF_v-FG_u)$$

$$=(\Gamma_{12}^1+\Gamma_{22}^2)\sqrt{g}.\quad(3)$$

由于 $\frac{\partial}{\partial u^l}g_{ij}=\boldsymbol{r}_{il}\cdot\boldsymbol{r}_j+\boldsymbol{r}_i\cdot\boldsymbol{r}_{jk}$

$$=[il,j]+[jk,i]$$

$$=\sum_m\Gamma_{il}^m g_{mj}+\sum_m\Gamma_{jl}^m g_{mi},$$

所以 $\Gamma_{11}^2\frac{\partial}{\partial u^2}g_{11}=\Gamma_{11}^2(\Gamma_{12}^1g_{11}+\Gamma_{12}^2g_{21}+\Gamma_{12}^1g_{11}+\Gamma_{12}^2g_{21})$

$$=2(g_{11}\Gamma_{12}^1+g_{12}\Gamma_{12}^2)\Gamma_{11}^2.$$

同理 $\Gamma_{12}^2\frac{\partial}{\partial u^1}g_{11}=2(g_{11}\Gamma_{11}^1+g_{12}\Gamma_{12}^2)\Gamma_{12}^2.$

因此 $\Gamma_{11}^2\frac{\partial}{\partial u^2}g_{11}-\Gamma_{12}^2\frac{\partial}{\partial u^1}g_{11}=2g_{11}(\Gamma_{11}^1\Gamma_{12}^2-\Gamma_{12}^1\Gamma_{11}^2),\quad(4)$

将(2)、(3)、(4)式代入(1),

原式右边$=\frac{1}{\sqrt{g}}\left[\frac{\sqrt{g}}{(g_{11})^2}\cdot2g_{11}(\Gamma_{11}^1\Gamma_{12}^2-\Gamma_{12}^1\Gamma_{12}^2)+\right.$

(3) 证明　由于 $K=-\frac{R_{1212}}{g}$,所以

$$R^1_{212}=g^{12}R_{2121}=g^{11}R_{1212}=g^{11}R_{1212}=-g^{11}gK$$
$$=-g_{22}K.$$

又因 $$R^1_{212}=\frac{\partial}{\partial v}\Gamma^1_{12}-\frac{\partial}{\partial u}\Gamma^1_{22}+\Gamma^2_{21}\Gamma^1_{22}-\Gamma^2_{22}\Gamma^1_{21},$$

所以

$$g_{22}K=\frac{\partial}{\partial u}\Gamma^1_{22}-\frac{\partial}{\partial v}\Gamma^1_{12}+\Gamma^1_{22}\Gamma^1_{11}+\Gamma^2_{22}\Gamma^1_{21}-\Gamma^1_{21}\Gamma^1_{12}-\Gamma^2_{21}\Gamma^1_{22}$$
$$=(\Gamma^1_{22})_u-(\Gamma^1_{12})_v+\Gamma^1_{22}(\Gamma^1_{11}+\Gamma^2_{12})-$$

$$\frac{1}{g_{11}}(-\sqrt{g}\Gamma^1_{11}\Gamma^2_{12}-\sqrt{g}\Gamma^2_{12}\Gamma^2_{12}+\sqrt{g}\Gamma^1_{12}\Gamma^2_{11}+\sqrt{g}\Gamma^2_{22}\Gamma^2_{11})+$$
$$\frac{\sqrt{g}}{g_{11}}\left(\frac{\partial}{\partial u^2}\Gamma^2_{12}-\frac{\partial}{\partial u^1}\Gamma^2_{12}\right)]$$
$$=\frac{1}{g_{11}}[(\Gamma^2_{11})_v-(\Gamma^2_{12})_u+\Gamma^2_{11}(\Gamma^2_{22}+\Gamma^1_{12})-$$
$$\Gamma^2_{12}(\Gamma^2_{12}+\Gamma^1_{11})+2(\Gamma^1_{11}\Gamma^2_{12}-\Gamma^1_{12}\Gamma^2_{12})]$$

另一方面,由于

$$K=-\frac{R_{1212}}{g},$$

所以 $$R^2_{121}=g^{22}R_{2121}=g^{22}R_{2121}=-g^{22}gK=-g_{11}K.$$

因为 $$R^2_{121}=\frac{\partial}{\partial u^1}\Gamma^2_{12}-\frac{\partial}{\partial u^2}\Gamma^2_{11}+\Gamma^2_{12}\Gamma^2_{21}-\Gamma^2_{11}\Gamma^2_{22},$$

故 $$K=-\frac{1}{g_{11}}[(\Gamma^2_{12})_u-(\Gamma^2_{11})_v+\Gamma^1_{12}\Gamma^2_{11}+\Gamma^2_{12}\Gamma^2_{21}-$$
$$\Gamma^1_{11}\Gamma^2_{12}-\Gamma^2_{11}\cdot\Gamma^2_{22}]$$
$$=\frac{1}{g_{11}}[(\Gamma^2_{11})_v-(\Gamma^2_{12})_u+\Gamma^2_{11}(\Gamma^2_{22}+\Gamma^1_{12})-$$
$$\Gamma^2_{12}(\Gamma^2_{12}+\Gamma^1_{11})+2(\Gamma^1_{11}\Gamma^2_{12}-\Gamma^2_{11}\Gamma^1_{12})].$$

$$\Gamma_{21}^{1}(\Gamma_{22}^{2}+\Gamma_{12}^{1})+2(\Gamma_{21}^{1}\Gamma_{22}^{2}-\Gamma_{22}^{1}\Gamma_{12}^{2}). \tag{1}$$

因为

$$\begin{cases}\Gamma_{22}^{2}+\Gamma_{12}^{1}=\dfrac{1}{\sqrt{EG-F^2}}\dfrac{\partial}{\partial v}\sqrt{EG-F^2},\\ \Gamma_{12}^{2}+\Gamma_{11}^{1}=\dfrac{1}{\sqrt{EG-F^2}}\dfrac{\partial}{\partial u}\sqrt{EG-F^2},\end{cases} \tag{2}$$

又因

$$\begin{cases}\dfrac{\partial}{\partial v}g_{22}=2g_{22}\Gamma_{22}^{2}=2(g_{21}\Gamma_{22}^{1}+g_{22}\Gamma_{22}^{2}),\\ \dfrac{\partial}{\partial u}g_{22}=2g_{22}\Gamma_{21}^{1}=2(g_{21}\Gamma_{21}^{1}+g_{22}\Gamma_{21}^{2}),\end{cases}$$

消去 g_{21}，得

$$\Gamma_{21}^{1}\frac{\partial}{\partial v}g_{22}-\Gamma_{22}^{1}\frac{\partial}{\partial u}g_{22}=2g_{22}(\Gamma_{22}^{2}\Gamma_{21}^{1}-\Gamma_{21}^{2}\Gamma_{22}^{1}), \tag{3}$$

将(2)、(3)代入(1)式，得

$$g_{22}K=\frac{\partial\Gamma_{22}^{1}}{\partial u}-\frac{\partial\Gamma_{21}^{1}}{\partial v}+\Gamma_{22}^{1}\frac{1}{\sqrt{EG-F^2}}\frac{\partial\sqrt{EG-F^2}}{\partial u}-$$

$$\Gamma_{21}^{1}\frac{1}{\sqrt{EG-F^2}}\frac{\partial\sqrt{EG-F^2}}{\partial v}+\frac{1}{g_{22}}\left(\Gamma_{21}^{1}\frac{\partial g_{22}}{\partial v}-\Gamma_{22}^{1}\frac{\partial g_{22}}{\partial u}\right).$$

所以 $$K=\frac{1}{g_{22}\sqrt{EG-F^2}}\left\{\frac{\partial\Gamma_{22}^{1}}{\partial u}\sqrt{EG-F^2}-\frac{\partial\Gamma_{21}^{1}}{\partial v}\sqrt{EG-F^2}\right\}-$$

$$\frac{1}{(g_{22})^2\sqrt{EG-F^2}}\left\{\Gamma_{22}^{1}\sqrt{EG-F^2}\frac{\partial g_{22}}{\partial u}-\Gamma_{21}^{1}\sqrt{EG-F^2}\frac{\partial g_{22}}{\partial v}\right\}$$

$$=\frac{1}{\sqrt{EG-F^2}}\left[\frac{\partial}{\partial u}\left(\frac{\sqrt{EG-F^2}}{G}\Gamma_{22}^{1}\right)-\frac{\partial}{\partial v}\left(\frac{\sqrt{EG-F^2}}{G}\Gamma_{12}^{1}\right)\right].$$

(4) 证明　因为 $ds^2=\lambda^2(du^2+dv^2)$，所以

$E=\lambda^2,\quad F=0,\quad G=\lambda^2,$

$E_u=2\lambda\lambda_u, E_v=2\lambda\lambda_v, G_u=2\lambda\lambda_u, G_v=2\lambda\lambda_v.$

因为 $F=0$，曲纹坐标网是正交网，所以

$$\Gamma_{11}^{1}=\frac{E_u}{2E}=\frac{\lambda_u}{\lambda},\qquad \Gamma_{11}^{2}=\frac{-E_v}{2G}=-\frac{\lambda_v}{\lambda},$$
$$\Gamma_{12}^{1}=\frac{E_v}{2E}=\frac{\lambda_v}{\lambda},\qquad \Gamma_{12}^{2}=\frac{G_u}{2G}=\frac{\lambda_u}{\lambda},$$
$$\Gamma_{22}^{1}=\frac{-G_u}{2E}=\frac{-\lambda_u}{\lambda},\qquad \Gamma_{22}^{2}=\frac{G_v}{2G}=\frac{\lambda_v}{\lambda}.$$

将 Γ_{ij}^{k} 代入下式

$$K=\frac{1}{E}\left[(\Gamma_{11}^{2})_v-(\Gamma_{12}^{2})_u+\Gamma_{11}^{1}\Gamma_{12}^{2}+\Gamma_{11}^{2}\Gamma_{22}^{2}-\Gamma_{12}^{1}\Gamma_{11}^{2}-(\Gamma_{12}^{2})^2\right]$$
$$=\frac{1}{\lambda^2}\left[\left(\frac{-\lambda_v}{\lambda}\right)_v-\left(\frac{\lambda_u}{\lambda}\right)_u+\frac{\lambda_u^2}{\lambda^2}+\frac{-\lambda_v^2}{\lambda^2}-\frac{-\lambda_v^2}{\lambda^2}-\frac{\lambda_u^2}{\lambda^2}\right]$$
$$=-\frac{1}{\lambda^2}\left[\left(\frac{\lambda_v}{\lambda}\right)_v+\left(\frac{\lambda_u}{\lambda}\right)_u\right]$$
$$=-\frac{1}{\lambda^2}\left[(\ln\lambda)_{uu}+(\ln\lambda)_{vv}\right].$$

(5) 证明　由于 $ds^2=du^2+Gdv^2$，所以

$$E=1,F=0,G=G.$$
$$\Gamma_{11}^{1}=\frac{E_u}{2E}=0,\Gamma_{11}^{2}=\frac{-E_v}{2G}=0,\Gamma_{12}^{1}=\frac{E_v}{2E}=0,$$
$$\Gamma_{12}^{2}=\frac{G_u}{2G},\Gamma_{22}^{1}=\frac{-G_u}{2E}=\frac{-G_u}{2},\Gamma_{22}^{2}=\frac{G_v}{2G}.$$

将 Γ_{ij}^{k} 代入下式，得

$$K=\frac{1}{E}\left\{(\Gamma_{11}^{2})_v-(\Gamma_{12}^{2})_u+\Gamma_{11}^{1}\Gamma_{12}^{2}+\Gamma_{11}^{2}\Gamma_{22}^{2}-\Gamma_{12}^{1}\Gamma_{11}^{2}-(\Gamma_{12}^{2})^2\right\}$$
$$=\left\{-\left(\frac{G_u}{2G}\right)_u-\left(\frac{G_u}{2G}\right)^2\right\}$$
$$=\frac{G_u^2-2GG_{uu}}{4G^2}.$$

因为　$(\sqrt{G})_u=\frac{1}{2}G^{-\frac{1}{2}}G_u$，

$$(\sqrt{G})_{uu}=\frac{1}{2}\left(-\frac{1}{2}\right)G^{-\frac{3}{2}}G_u^2+\frac{1}{2}G^{-\frac{1}{2}}G_{uu}$$

$$= -\frac{1}{4}\frac{G_u^2}{\sqrt{G^3}} + \frac{1}{2}\frac{GG_{uu}}{\sqrt{G^3}},$$

所以 $K = -\frac{1}{\sqrt{G}}\frac{\partial^2 \sqrt{G}}{\partial u^2}$.

7 解法一 因为

$$E = \frac{1}{(u^2+v^2+C)^2}, F = 0, G = \frac{1}{(u^2+v^2+C)^2}.$$

所以 $E_u = \frac{-2(u^2+v^2+C)\cdot 2u}{(u^2+v^2+C)^4} = \frac{-4u}{(u^2+v^2+C)^3} = G_u$,

$$E_v = \frac{-2(u^2+v^2+C)\cdot 2v}{(u^2+v^2+C)^4} = \frac{-4v}{(u^2+v^2+C)^3} = G_v,$$

所以 $\Gamma_{11}^1 = \frac{E_u}{2E} = \frac{-2u}{u^2+v^2+C}, \Gamma_{11}^2 = \frac{-E_v}{2G} = \frac{2v}{u^2+v^2+C}$,

$$\Gamma_{12}^1 = \frac{E_v}{2E} = \frac{-2v}{u^2+v^2+C}, \Gamma_{12}^2 = \frac{G_u}{2G} = \frac{-2u}{u^2+v^2+C},$$

$$\Gamma_{22}^1 = \frac{-G_u}{2E} = \frac{2u}{u^2+v^2+C}, \Gamma_{22}^2 = \frac{G_v}{2G} = \frac{-2v}{u^2+v^2+C}.$$

解法二 因为 $g_{11} = g_{22} = \frac{1}{(u^2+v^2+C)^2}, g_{12} = g_{21} = 0$.

$$[ij,l] = \frac{1}{2}\left(\frac{\partial g_{il}}{\partial u^j} + \frac{\partial g_{jl}}{\partial u^i} - \frac{\partial g_{ij}}{\partial u^l}\right) = \sum_l \Gamma_{ij}^k g_{kl}.$$

所以 $\Gamma_{ij}^k = \sum_l g^{kl}[ij,l]$

$$= \sum_l \frac{1}{2} g^{kl}\left(\frac{\partial g_{il}}{\partial u^j} + \frac{\partial g_{jl}}{\partial u^i} - \frac{\partial g_{ij}}{\partial u^l}\right),$$

所以 $\Gamma_{11}^1 = \sum_l g^{1l}[11,l] = \sum_l \frac{1}{2} g^{1l}\left(\frac{\partial g_{1l}}{\partial u^1} + \frac{\partial g_{1l}}{\partial u^1} - \frac{\partial g_{11}}{\partial u^l}\right)$

$$= \frac{1}{2} g^{11}\left(\frac{\partial g_{11}}{\partial u^1} + \frac{\partial g_{11}}{\partial u^1} - \frac{\partial g_{11}}{\partial u^1}\right) + \frac{1}{2} g^{12}\left(\frac{\partial g_{12}}{\partial u^1} + \frac{\partial g_{12}}{\partial u^1} - \frac{\partial g_{11}}{\partial u^2}\right)$$

$$= \frac{1}{2}\frac{g_{22}}{g}\frac{\partial g_{11}}{\partial u^1} - \frac{1}{2}\frac{-g_{12}}{g}\left(\frac{\partial g_{12}}{\partial u^1} + \frac{\partial g_{12}}{\partial u^1} - \frac{\partial g_{11}}{\partial u^2}\right)$$

$$= \frac{1}{2} \cdot \frac{\frac{1}{(u^2+v^2+C)^2}}{\frac{1}{(u^2+v^2+C)^4}} \cdot \frac{-4u}{(u^2+v^2+C)^3}$$

$$= -\frac{2u}{(u^2+v^2+C)}.$$

同理可得其余各量.

8 证明 由上题知

$$\Gamma_{11}^1 = \frac{-2u}{(u^2+v^2+C)}, \quad \Gamma_{11}^2 = \frac{2v}{(u^2+v^2+C)},$$

$$\Gamma_{12}^1 = \frac{-2v}{(u^2+v^2+C)}, \quad \Gamma_{12}^2 = \frac{-2u}{(u^2+v^2+C)},$$

$$\Gamma_{22}^1 = \frac{-2v}{(u^2+v^2+C)}, \quad \Gamma_{22}^2 = \frac{-2v}{(u^2+v^2+C)}.$$

所以 $$(\Gamma_{11}^2)_v = \frac{2(u^2+v^2+C)-4v^2}{(u^2+v^2+C)^2} = \frac{2u^2+2v^2+2C}{(u^2+v^2+C)^2},$$

$$(\Gamma_{12}^2)_u = \frac{-2(u^2+v^2+C)+4u^2}{(u^2+v^2+C)^2} = \frac{2u^2-2v^2-2C}{(u^2+v^2+C)^2},$$

$$\Gamma_{11}^1\Gamma_{12}^2 = \frac{2u^2}{(u^2+v^2+C)^2}, \Gamma_{11}^2\Gamma_{22}^2 = \frac{-4v^2}{(u^2+v^2+C)^2},$$

$$\Gamma_{12}^1\Gamma_{11}^2 = \frac{-4v^2}{(u^2+v^2+C)^2}, \Gamma_{12}^2\Gamma_{12}^2 = \frac{4u^2}{(u^2+v^2+C)^2},$$

所以

$$K = \frac{1}{E}\{(\Gamma_{11}^2)_v - (\Gamma_{12}^2)_u + \Gamma_{11}^1\Gamma_{12}^2 + \Gamma_{11}^2\Gamma_{22}^2 - \Gamma_{12}^1\Gamma_{11}^2 - \Gamma_{12}^2\Gamma_{12}^2\}$$

$$= \frac{(u^2+v^2+C)^2}{1} \cdot \frac{4C}{(u^2+v^2+C)^2} = 4C = \text{const}.$$

9 解 已知 $E = G = 1, F = 0, L = -1, N = M = 0$.

先求第二类克氏符号:

$$\Gamma_{11}^1 = \frac{E_u}{2E} = 0, \qquad \Gamma_{11}^2 = -\frac{E_v}{2G} = 0,$$

$$\Gamma_{12}^1=\frac{E_v}{2E}=0,\qquad \Gamma_{12}^2=\frac{G_u}{2G}=0,$$

$$\Gamma_{22}^1=-\frac{G_u}{2E}=0,\quad \Gamma_{22}^2=\frac{G_v}{2G}=0.$$

再计算 μ_i^j：

$$\mu_1^1=-\frac{L}{E}=1,\qquad \mu_1^2=-\frac{M}{G}=0,$$

$$\mu_2^1=-\frac{M}{E}=0,\qquad \mu_2^2=-\frac{N}{G}=0.$$

可以验证它们满足高斯－科达齐方程.于是所求平面存在，根据高斯－魏因加因公式有：

$$\boldsymbol{r}_{uu}=-\boldsymbol{n},\tag{1}$$

$$\boldsymbol{r}_{uv}=\boldsymbol{0},\tag{2}$$

$$\boldsymbol{r}_{vv}=\boldsymbol{0},\tag{3}$$

$$\boldsymbol{n}_u=\boldsymbol{r}_u,\tag{4}$$

$$\boldsymbol{n}_v=\boldsymbol{0}.\tag{5}$$

由方程(1)与(4)得

$$\boldsymbol{r}_{uuu}+\boldsymbol{r}_u=\boldsymbol{0}.$$

积分得

$$\boldsymbol{r}=\boldsymbol{e}_1(v)\sin u+\boldsymbol{e}_2(v)\cos u+\boldsymbol{e}_3(v).$$

于是

$$\boldsymbol{r}_u=\boldsymbol{e}_1(v)\cos u-\boldsymbol{e}_2(v)\sin u,$$

$$\boldsymbol{r}_{uv}=\boldsymbol{e}'_1(v)\cos u-\boldsymbol{e}'_2(v)\sin u.$$

根据上式与方程(2)知

$$\boldsymbol{e}'_1(v)\cos u-\boldsymbol{e}'_2(v)\sin u=0,$$

即

$$\boldsymbol{e}'_1(v)=\boldsymbol{e}'_2(v)\tan u,$$

由于 $\boldsymbol{e}_1(v)$，$\boldsymbol{e}_2(v)$ 只与 v 有关，故上式成立当且仅当 $\boldsymbol{e}'_1(v)=\boldsymbol{e}'_2(v)=\boldsymbol{0}$，所以 $\boldsymbol{e}_1(v)$、$\boldsymbol{e}_2(v)$ 是常向量.

$$r = e_1 \sin u + e_2 \cos u + e_3(v).$$

由此得

$$r_v = e'_3(v), r_{vv} = e''_3(v),$$

根据方程(3) $\qquad r_{vv} = e''_3(v) = 0,$

故有 $\qquad e_3(v) = a + bv$ （其中 a, b 是常向量）.

所求曲面的方程为

$$r = e_1 \sin u + e_2 \cos u + (a + bv).$$

10 证明 由于 $E = G = 1, F = 0, L = 1, M = 0, N = -1$,所以

$$\Gamma_{11}^1 = \Gamma_{11}^2 = \Gamma_{12}^1 = \Gamma_{12}^2 = \Gamma_{22}^1 = \Gamma_{22}^2 = 0,$$

$$\mu_1^1 = -\frac{L}{E} = -1, \qquad \mu_1^2 = -\frac{M}{G} = 0,$$

$$\mu_2^1 = -\frac{M}{E} = 0, \qquad \mu_2^2 = -\frac{N}{G} = 1.$$

由上可知 $R_{m\gamma k} = 0$ 特别 $R_{1212} = 0$. 但是

$$R_{1212} = L_{21}L_{12} - L_{22}L_{11}, \text{(高斯公式)}$$

$$L_{21}L_{12} - L_{22}L_{11} = M^2 - LN = 1 \neq 0,$$

不满足高斯公式,故曲面不存在.

习题 2.6

1 解 因为是正交网,所以有

$$F = 0, \quad \Gamma_{22}^1 = -\frac{G_u}{2E}, \quad \Gamma_{11}^2 = \frac{-E_v}{2G}.$$

对于 v－曲线, $\mathrm{d}u = 0$,所以

$$\frac{\mathrm{d}v}{\mathrm{d}s} = \frac{\mathrm{d}v}{\sqrt{G}\mathrm{d}v} = \frac{1}{\sqrt{G}},$$

$$k_g = \sqrt{EG}\left[-\frac{\mathrm{d}v}{\mathrm{d}s}\left(\Gamma_{22}^1 \frac{\mathrm{d}v}{\mathrm{d}s} \cdot \frac{\mathrm{d}v}{\mathrm{d}s}\right)\right]$$

$$= \sqrt{EG}\left(\frac{\mathrm{d}v}{\mathrm{d}s}\right)^3 \cdot \frac{G_u}{2E}$$

$$= \frac{G_u}{2G\sqrt{E}}.$$

对于 u－曲线，$\mathrm{d}v=0$，

$$\frac{\mathrm{d}u}{\mathrm{d}s}=\frac{1}{\sqrt{E}},$$

$$k_g=\sqrt{EG}\left[\frac{\mathrm{d}u}{\mathrm{d}s}\cdot\Gamma_{11}^2\frac{\mathrm{d}u}{\mathrm{d}s}\cdot\frac{\mathrm{d}u}{\mathrm{d}s}\right]$$

$$=-\frac{E_v}{2E\sqrt{G}}.$$

2　解　球面 $\boldsymbol{r}=\{a\cos u\cos v, a\cos u\sin v, a\sin u\}$，

$$G=a^2\cos^2 u, F=0, E=a^2,$$

$$\mathrm{I}=a^2\cos^2 u\mathrm{d}v^2+a^2\mathrm{d}u^2.$$

设 θ 是曲面上曲线与 u－曲线方向 $\boldsymbol{e}_1$ 的夹角. 根据测地面率的刘维尔公式：

$$k_g=\frac{\mathrm{d}\theta}{\mathrm{d}s}-\frac{E_v}{2\sqrt{EG}}\frac{\mathrm{d}u}{\mathrm{d}s}+\frac{G_u}{2\sqrt{EG}}\frac{\mathrm{d}v}{\mathrm{d}s}$$

$$=\frac{\mathrm{d}\theta}{\mathrm{d}s}-\frac{2a^2\cos u\sin u}{2a^2\sqrt{\cos^2 u}}\frac{\mathrm{d}v}{\mathrm{d}s}$$

$$=\frac{\mathrm{d}\theta}{\mathrm{d}s}-\frac{\sin u\mathrm{d}v}{\mathrm{d}s}.$$

3　解　设球面为 $\boldsymbol{r}=\{R\cos u\cos v, R\cos u\sin v, R\sin u\}$，

$$E=R^2,\quad F=0,\quad G=R^2\cos^2 u.$$

半径为 a 的圆 (C) 的单位切向量为 $\boldsymbol{\alpha}$，它与经线的夹角 $\theta=-\frac{\pi}{2}$，根据刘维尔公式有

$$\frac{\mathrm{d}v}{\mathrm{d}s}=\frac{1}{\sqrt{G}}\sin\theta=\frac{1}{R\cos u}\sin\left(-\frac{\pi}{2}\right)=\frac{-1}{R\cos u}.$$

代入第二题结果，得

$$k_g=\frac{1}{R}\tan u=\frac{\sqrt{R^2-a^2}}{Ra}.$$

4 解 因为正螺面的第一基本形式为

$$\mathrm{I}=\mathrm{d}u^2+(u^2+a^2)\mathrm{d}v^2.$$

螺旋线是正螺面的 v－曲线 $u=u_0$，由第一题结果可得

$$k_g=\frac{G_u}{2G\sqrt{E}}=\frac{u_0}{u_0^2+a^2}.$$

5 证明 设曲线 (C) 的球面象为 $(\bar{C})$．(C) 与 $(\bar{C})$ 的测地曲率分别为 k_g 和 $\bar{k}_g$．

由题设 (C) 为 S 上的曲率线，根据罗德里格定理 $\mathrm{d}\boldsymbol{n}=-k_n\mathrm{d}\boldsymbol{r}$，于是 $(\bar{C})$ 的切向量

$$\bar{\boldsymbol{\alpha}}=\frac{\mathrm{d}\boldsymbol{n}}{\mathrm{d}\bar{s}}=\frac{-k_n\mathrm{d}\boldsymbol{r}}{\mathrm{d}\bar{s}}=-k_n\frac{\mathrm{d}s}{\mathrm{d}\bar{s}}\boldsymbol{\alpha}.$$

即

$$\bar{\boldsymbol{\alpha}}=\delta\boldsymbol{\alpha},\ \delta=-k_n\frac{\mathrm{d}s}{\mathrm{d}\bar{s}}=\pm1.$$

在球面上，不妨取 $\bar{\boldsymbol{n}}=\boldsymbol{n}$．$\bar{\boldsymbol{\varepsilon}}=\boldsymbol{n}\times\boldsymbol{\alpha}=\delta\boldsymbol{\varepsilon}$．

$$\begin{aligned}\bar{k}_g&=\dot{\bar{\boldsymbol{\alpha}}}\cdot\bar{\boldsymbol{\varepsilon}}=\delta^2\dot{\boldsymbol{\alpha}}\boldsymbol{\varepsilon}\frac{\mathrm{d}s}{\mathrm{d}\bar{s}}\\&=k_g\cdot\left(-\frac{\delta}{k_n}\right)=-\delta\frac{k_g}{k_n}.\end{aligned}$$

所以

$$|k_g|=|k_n\cdot\bar{k}_g|.$$

6 解 设曲面曲线的方程为 $u=u(t)$，$v=v(t)$，则

$$\frac{\mathrm{d}u}{\mathrm{d}s}=\frac{\mathrm{d}u}{\mathrm{d}t}\cdot\frac{\mathrm{d}t}{\mathrm{d}s},\ \frac{\mathrm{d}^2u}{\mathrm{d}s^2}=\frac{\mathrm{d}^2u}{\mathrm{d}t^2}\left(\frac{\mathrm{d}t}{\mathrm{d}s}\right)^2+\frac{\mathrm{d}u}{\mathrm{d}t}\cdot\frac{\mathrm{d}^2t}{\mathrm{d}s^2}.$$

代入测地曲率的计算公式

$$\begin{aligned}k_g&=\sqrt{g}\left[\frac{\mathrm{d}u^1}{\mathrm{d}s}\left(\frac{\mathrm{d}^2u^2}{\mathrm{d}s^2}+\sum_{ij}\Gamma_{ij}^2\frac{\mathrm{d}u^i}{\mathrm{d}s}\cdot\frac{\mathrm{d}u^j}{\mathrm{d}s}\right)-\right.\\&\qquad\left.\frac{\mathrm{d}u^2}{\mathrm{d}s}\left(\frac{\mathrm{d}^2u^1}{\mathrm{d}s^2}+\sum_{ij}\Gamma_{ij}^1\frac{\mathrm{d}u^i}{\mathrm{d}s}\cdot\frac{\mathrm{d}u^j}{\mathrm{d}s}\right)\right]\\&=\sqrt{g}\left[\frac{\mathrm{d}u^1}{\mathrm{d}t}\cdot\frac{\mathrm{d}t}{\mathrm{d}s}\left(\frac{\mathrm{d}^2u^2}{\mathrm{d}t^2}\left(\frac{\mathrm{d}t}{\mathrm{d}s}\right)^2+\frac{\mathrm{d}u^2}{\mathrm{d}t}\left(\frac{\mathrm{d}^2t}{\mathrm{d}s^2}\right)+\right.\right.\\&\qquad\left.\sum_{ij}\Gamma_{ij}^2\frac{\mathrm{d}u^i}{\mathrm{d}t}\cdot\frac{\mathrm{d}u^j}{\mathrm{d}t}\left(\frac{\mathrm{d}t}{\mathrm{d}s}\right)^2\right)-\frac{\mathrm{d}u^2}{\mathrm{d}t}\cdot\frac{\mathrm{d}t}{\mathrm{d}s}\left(\frac{\mathrm{d}^2u^1}{\mathrm{d}t^2}\left(\frac{\mathrm{d}t}{\mathrm{d}s}\right)^2+\right.\end{aligned}$$

$$\frac{\mathrm{d}u^1}{\mathrm{d}t}\left(\frac{\mathrm{d}^2 t}{\mathrm{d}s^2}\right)+\sum_{ij}\Gamma^1_{ij}\frac{\mathrm{d}u^i}{\mathrm{d}t}\frac{\mathrm{d}u^j}{\mathrm{d}t}\left(\frac{\mathrm{d}t}{\mathrm{d}s}\right)^2\Big)\Big].$$

7 证明 (1) 设子午线为 $\boldsymbol{r}=\boldsymbol{r}(s)$，由于 $\boldsymbol{\beta},\boldsymbol{\alpha},\boldsymbol{n}$ 共面，且 $\boldsymbol{n}\perp\boldsymbol{\alpha},\boldsymbol{\beta}\perp\boldsymbol{\alpha}$，所以

$$\boldsymbol{n}\,/\!/\,\boldsymbol{\beta}.$$

故子午线 $\boldsymbol{r}=\boldsymbol{r}(s)$ 是测地线.

(2) 设子午线的切向量为 $\boldsymbol{\alpha}_0$. 平行圆的主法向量为 $\boldsymbol{\beta}$. 若 $\boldsymbol{\alpha}_0$ 平行于旋转轴，则因 $\boldsymbol{\alpha}_0,\boldsymbol{\beta},\boldsymbol{n}$ 共面且 $\boldsymbol{\alpha}_0\perp\boldsymbol{\beta},\boldsymbol{\alpha}_0\perp\boldsymbol{n}$，故有 $\boldsymbol{n}\,/\!/\,\boldsymbol{\beta}$，即平行圆当子午线的切线平行于旋转轴时，是测地线. 反之也成立.

8 证明

(1) 由 $k^2=k_g^2+k_n^2=0$，可知 $k=0$，所以曲线为直线.

(2) 设 (C) 为测地线，又是曲率线，则当 (C) 是直线时，当然 (C) 是平面曲线. 当 (C) 不是直线时，由

$$\boldsymbol{\beta}=\pm\boldsymbol{n}\quad(\text{根据}(C)\text{是测地线})$$

可知 $-k\boldsymbol{\alpha}+\tau\boldsymbol{\gamma}=\pm\dot{\boldsymbol{n}}=\pm\lambda\boldsymbol{\alpha}$（根据 (C) 是曲率线，依罗德里格定理）. 所以 $\tau=0$，即 (C) 是平面曲线.

9 解 由于 $\mathrm{d}s^2=v(\mathrm{d}u^2+\mathrm{d}v^2)$，所以有 $E=G=v,F=0$，于是有

$$E_v=G_v=1,G_u=E_u=0.$$

由测地线的微分方程得

$$\begin{cases}\dfrac{\mathrm{d}u}{\mathrm{d}s}=\dfrac{1}{\sqrt{E}}\cos\theta=\dfrac{1}{\sqrt{v}}\cos\theta,\\[2mm]\dfrac{\mathrm{d}v}{\mathrm{d}s}=\dfrac{1}{\sqrt{G}}\sin\theta=\dfrac{1}{\sqrt{v}}\sin\theta,\\[2mm]\dfrac{\mathrm{d}\theta}{\mathrm{d}s}=\dfrac{1}{2\sqrt{GE}}\left(\dfrac{E_v}{\sqrt{E}}\cos\theta-\dfrac{G_u}{\sqrt{G}}\sin\theta\right)\\[2mm]\qquad=\dfrac{1}{2v\sqrt{v}}\cos\theta=\dfrac{1}{2v}\dfrac{\mathrm{d}u}{\mathrm{d}s}.\end{cases}$$

由前两个方程得

$$\sin\theta \mathrm{d}u = \cos\theta \mathrm{d}v, \quad 即 \quad \frac{\mathrm{d}u}{\mathrm{d}v} = \cot\theta.$$

由后一个方程得

$$\mathrm{d}\theta = \frac{1}{2v}\mathrm{d}u,$$

$$\sin\theta \mathrm{d}\theta = \sin\theta \cdot \frac{1}{2v}\mathrm{d}u = \sin\theta \cdot \frac{1}{2v}\cot\theta \mathrm{d}v,$$

$$= \frac{1}{2v}\cos\theta \mathrm{d}v,$$

则有

$$\tan\theta \mathrm{d}\theta = \frac{1}{2v}\mathrm{d}v.$$

两边积分得

$$\sqrt{v}\cos\theta = C, \cos\theta = \frac{C}{\sqrt{v}} (C \text{ 为积分常数}),$$

$$\frac{\mathrm{d}u}{\mathrm{d}v} = \cot\theta = \frac{\cos\theta}{\sqrt{1-\cos^2\theta}} = \frac{C}{\sqrt{v - C^2}},$$

所以

$$\mathrm{d}u = \frac{C\mathrm{d}v}{\sqrt{v - C^2}}.$$

积分得

$$u = 2C\sqrt{v - C^2},$$

即

$$u^2 = 4C^2(v - C^2).$$

所求的测地线在 uv 平面上是抛物线.

10 解 由于 $\boldsymbol{r} = \{u\cos v, u\sin v, av\}$,

$$E = 1, \quad F = 0, \quad G = u^2 + a^2.$$

由测地线的微分方程得

$$\frac{\mathrm{d}u}{\mathrm{d}v} = \sqrt{\frac{E}{G}}\tan\theta = \frac{1}{\sqrt{u^2 + a^2}}\tan\theta. \tag{1}$$

$$\frac{d\theta}{du}=\frac{1}{2}\sqrt{\frac{E}{G}}\frac{\partial \ln E}{\partial v}-\frac{1}{2}\frac{\partial \ln G}{\partial u}\tan\theta$$
$$=-\frac{1}{2}\frac{\partial \ln(u^2+a^2)}{\partial u}\tan\theta$$
$$=-\frac{1}{2}\frac{d\ln(u^2+a^2)}{du}\tan\theta, \tag{2}$$

由(2)式得

$$\ln\sin\theta=\frac{1}{2}\ln(u^2+a^2)+\ln C,$$

$$\sin\theta=\frac{C}{\sqrt{u^2+a^2}},$$

$$\cos\theta=\frac{\sqrt{u^2+a^2-C^2}}{\sqrt{u^2+a^2}}.$$

再根据(1)式得

$$dv=\frac{C du}{\sqrt{u^2+a^2-C^2}\sqrt{u^2+a^2}},$$

$$v=C\int_{u_0}^{u}\frac{du}{\sqrt{u^2+a^2-C^2}\sqrt{u^2+a^2}},$$

此即测地线的曲纹坐标表示.

11 证明 (1) 对于平面 $ds^2=du^2+dv^2$,

$$E=G=1,\quad F=0,\quad E_u=E_v=G_u=G_v=0.$$

代入测地线的方程得

$$\frac{d\theta}{ds}=0,\quad \theta=\text{const}.$$

$$\frac{du}{ds}=\cos\theta,\quad \frac{dv}{ds}=\sin\theta,\quad \frac{du}{dv}=\cot\theta=\text{const},$$

$$u=av+b,$$

是平面上的直线.

(2) 对于圆柱面 $\boldsymbol{r}=\{R\cos\theta,R\sin\theta,z\}$,

$$E = R^2, \quad F = 0, \quad G = 1,$$
$$E_u = G_u = E_v = G_v = 0.$$

代入测地线的方程,解得

$$z = a\theta + b.$$

测地线为

$$\boldsymbol{r} = \{R\cos\theta, R\sin\theta, a\theta + b\},$$

是圆柱螺线.

12 证明 因为 $k_g = \pm k\sin\theta$, θ 是 $\boldsymbol{\beta}$ 与 $\boldsymbol{n}$ 的夹角, $k_g = 0$, $k \neq 0$, 所以 $\sin\theta = 0$, $\theta = 0$, 或 $\theta = \pi$, 即 $\boldsymbol{\beta} = \pm\boldsymbol{n}$.

$$\begin{aligned} \mathrm{d}\boldsymbol{n} &= \mathrm{d}\boldsymbol{\beta} = \dot{\boldsymbol{\beta}}\mathrm{d}s = (-k\boldsymbol{\alpha} + \tau\boldsymbol{\gamma})\mathrm{d}s \\ &= -k\boldsymbol{\alpha}\mathrm{d}s = -k\mathrm{d}\boldsymbol{r} \end{aligned}$$

根据罗德里格定理知测地线是曲率线.

13 证明 设曲面上曲线的方程为

$$u = u(t), \quad v = v(t).$$

因为

$$\begin{aligned} k_g &= \sqrt{g}\left[\frac{\mathrm{d}u^1}{\mathrm{d}s}\left(\frac{\mathrm{d}^2u^2}{\mathrm{d}s^2} + \sum_{ij}\Gamma_{ij}^2\frac{\mathrm{d}u^i}{\mathrm{d}s}\cdot\frac{\mathrm{d}u^j}{\mathrm{d}s}\right) - \right. \\ &\quad \left. \frac{\mathrm{d}u^2}{\mathrm{d}s}\left(\frac{\mathrm{d}^2u^1}{\mathrm{d}s^2} + \sum_{ij}\Gamma_{ij}^1\frac{\mathrm{d}u^i}{\mathrm{d}s}\frac{\mathrm{d}u^j}{\mathrm{d}s}\right)\right] \\ &= \frac{\sqrt{g}}{\mathrm{d}s^3}\left[u'\mathrm{d}t\left(v''\mathrm{d}t^2 + v'\mathrm{d}^2t + \frac{G_u}{2G}u'v'\mathrm{d}t^2 + \frac{G_u}{2G}u'v'\mathrm{d}t^2 + \right.\right. \\ &\quad \left.\left. \frac{G_v}{2G}v'^2\mathrm{d}t^2\right) - v'\mathrm{d}t\left(u''\mathrm{d}t^2 + u'\mathrm{d}^2t - \frac{G_u}{2}v'^2\mathrm{d}t^2\right)\right] \\ &= \frac{\sqrt{G}}{(u'^2 + Gv'^2)^{\frac{3}{2}}}\left[u'v'' - v'u'' + \frac{1}{2}G_uv'^3 + \frac{G_v}{2G}u'v'^2 + \right. \\ &\quad \left. \frac{G_u}{G}u'^2v'\right]. \end{aligned}$$

由于 $\mathrm{d}s = (u'^2 + Gv'^2)^{\frac{1}{2}}\mathrm{d}t$,

$$k_g\mathrm{d}s = \frac{\sqrt{G}}{(u'^2 + Gv'^2)}\left(u'v'' - u''v' + \frac{1}{2}G_uv'^2 + \frac{G_v}{2G}u'v'^2 + \right.$$

$$\frac{G_u}{G}u'^2v'\Big)dt. \tag{1}$$

另一方面，

$$d\left(\arctan\sqrt{G}\frac{dv}{du}\right)+\frac{\partial\sqrt{G}}{\partial u}dv=d\left(\arctan\sqrt{G}\frac{dv}{du}\right)+\frac{\partial\sqrt{G}}{\partial u}dv.$$

$$=\frac{\dfrac{\dfrac{G_u}{2\sqrt{G}}u'^2v'+\dfrac{G_v}{2\sqrt{G}}u'v'^2+\sqrt{G}u'v'-\sqrt{G}u''v'}{u'^2}}{1+\dfrac{Gv'^2}{u'^2}}dt+(\sqrt{G})_uv'dt$$

$$=\frac{\sqrt{G}u'v''-\sqrt{G}u''v'+\dfrac{G_u}{2\sqrt{G}}v'u'^2+\dfrac{G_v}{2\sqrt{G}}u'v'^2+\dfrac{G_u}{2\sqrt{G}}u'^2v+\dfrac{GG_u}{2\sqrt{G}}v'^3}{u'^2+Gv'^2}dt.$$

$$=\frac{\sqrt{G}}{(u'^2+Gv'^2)}\left(u'v''-u''v'+\frac{1}{2}G_uv'^3+\frac{G_u}{G}u'^2v'+\frac{G_v}{2G}u'v'^2\right)dt. \tag{2}$$

由(1)、(2)式得

$$k_g\,ds=d\left(\arctan\sqrt{G}\frac{dv}{du}\right)+\frac{\partial\sqrt{G}}{\partial u}dv.$$

14 证明 因为 $E=1, F=0, G=G(u,v)$，所以

$$\Gamma_{11}^1=0,\quad \Gamma_{12}^1=0,\quad \Gamma_{22}^1=-\frac{1}{2}G_u,$$

$$\Gamma_{11}^2=0,\quad \Gamma_{12}^2=\frac{G_u}{2G},\quad \Gamma_{22}^2=\frac{G_v}{2G}.$$

设测地线的方程为 $u=u(v)$，则它满足微分方程

$$\frac{d^2u^k}{ds^2}+\sum_{ij}\Gamma_{ij}^k\frac{du^i}{ds}\cdot\frac{du^j}{ds}=0\quad(k=1,2),$$

消去 ds，得

$$\frac{d^2u}{dv^2}=\Gamma_{11}^2\left(\frac{du}{dv}\right)^3+(2\Gamma_{21}^2-\Gamma_{11}^1)\left(\frac{du}{dv}\right)^2+$$

$$(\Gamma_{22}^2-2\Gamma_{12}^1)\frac{du}{dv}-\Gamma_{22}^1$$

$$= \frac{1}{G} G_u \left(\frac{\mathrm{d}u}{\mathrm{d}v}\right)^2 + \frac{1}{2G} G_v \frac{\mathrm{d}u}{\mathrm{d}v} + \frac{1}{2} G_u . \tag{1}$$

此测地线与 u－曲线的交角为 α 时，有

$$\cos\alpha = \frac{\dfrac{\mathrm{d}u}{\mathrm{d}v}}{\sqrt{\left(\dfrac{\mathrm{d}u}{\mathrm{d}v}\right)^2 + G}},$$

所以

$$\frac{\mathrm{d}u}{\mathrm{d}v} = \sqrt{G} \cot\alpha . \tag{2}$$

$$\frac{\mathrm{d}^2 u}{\mathrm{d}v^2} = -\sqrt{G} \csc^2\alpha \frac{\mathrm{d}\alpha}{\mathrm{d}v} + [(\sqrt{G})_v + (\sqrt{G})_u \sqrt{G} \cot\alpha] \cot\alpha . \tag{3}$$

将(2)代入(1)得

$$\frac{\mathrm{d}^2 u}{\mathrm{d}v^2} = G_u \cot^2\alpha + \frac{G_v}{2\sqrt{G}} \cot\alpha + \frac{1}{2} G_u . \tag{4}$$

将(4)代入(3)得

$$\sqrt{G} \csc^2\alpha \frac{\mathrm{d}\alpha}{\mathrm{d}v} + \frac{1}{2}(\cot^2\alpha + 1) G_u = 0 .$$

所以

$$\frac{\mathrm{d}\alpha}{\mathrm{d}v} = -(\sqrt{G})_u . \quad 即 \quad \frac{\mathrm{d}\alpha}{\mathrm{d}v} = -\frac{\partial\sqrt{G}}{\partial u} .$$

15 证明 在每族测地线中任取两条，围成曲面上的曲边四边形．根据已知条件，曲边四边形的外角和为 2π．由高斯－波涅公式有

$$\int_G K \mathrm{d}\sigma + 2\pi = 2\pi ,$$

$$\int_G K \mathrm{d}\sigma = 0 .$$

若在曲面的某点 P_0 处，$K \neq 0$，不妨设 $K(P_0) > 0$，则在 P_0 点邻近 $K > 0$，从面对于围绕 P_0 点的充分小的曲边四边形有

$$\int_G K\mathrm{d}\sigma > 0.$$

得出矛盾，所以 $K\equiv 0$，即曲面是可展面.

16 解　由高斯－波涅公式有

$$\iint_G K\mathrm{d}\sigma = S_{(\triangle)} - \pi.$$

对于半径为 R 的球面，$K=\dfrac{1}{R^2}$，所以

$$S_{(\triangle)} = \pi + \frac{1}{R^2}A_{(\triangle)},$$

其中 $A_{(\triangle)}$ 为测地三角形的面积.

17 解　与 15 题相同.

18 证明　设若存在所述闭测地线 (C). 它所围成的曲面部分为 G，则由高斯－波涅公式

$$\iint_G K\mathrm{d}\sigma + \oint_{\partial G} k_g \mathrm{d}s + \sum_{i=1}^{k}(\pi - \alpha_i) = 2\pi.$$

因为 $K<0$，则 $\iint_G K\mathrm{d}\sigma \leqslant 0$，又后两项均为 0，得出矛盾. 所以不存在所述闭测地线.

19 证明　设

$$\boldsymbol{a} = a^1\boldsymbol{r}_1 + a^2\boldsymbol{r}_2,$$

$$\boldsymbol{b} = b^1\boldsymbol{r}_1 + b^2\boldsymbol{r}_2.$$

则

$$\boldsymbol{a}+\boldsymbol{b} = (a^1+b^1)\boldsymbol{r}_1 + (a^2+b^2)\boldsymbol{r}_2,$$

$$f\boldsymbol{a} = fa^1\boldsymbol{r}_1 + fa^2\boldsymbol{r}_2,$$

$$\boldsymbol{a}\cdot\boldsymbol{b} = a^1b^1 + a^2b^2.$$

(1)　$\mathrm{D}(\boldsymbol{a}+\boldsymbol{b})$

$$= \mathrm{D}(a^1+b^1)\boldsymbol{r}_1 + \mathrm{D}(a^2+b^2)\boldsymbol{r}_2$$

$$= \Big(\mathrm{d}(a^1+b^1) + \sum_{\alpha,\beta=1}^{2}\Gamma^1_{\alpha\beta}(a^\alpha+b^\alpha)\mathrm{d}u^\beta\Big)\boldsymbol{r}_1 +$$

$$\left(\mathrm{d}(a^2+b^2)+\sum_{\alpha\beta=1}^{2}\Gamma_{\alpha\beta}^{2}(\theta^{\alpha}+b^{\alpha})\mathrm{d}u^{\beta}\right)\boldsymbol{r}_2$$

$$=\left(\mathrm{d}a^1+\sum_{\alpha\beta=1}^{2}\Gamma_{\alpha\beta}^{1}a^{\alpha}\mathrm{d}u^{\beta}\right)\boldsymbol{r}_1+$$

$$\left(\mathrm{d}a^2+\sum_{\alpha\beta=1}^{2}\Gamma_{\alpha\beta}^{2}a^{\alpha}\mathrm{d}u^{\beta}\right)\boldsymbol{r}_2+$$

$$\left(\mathrm{d}b^1+\sum_{\alpha\beta=1}^{2}\Gamma_{\alpha\beta}^{1}b^{\alpha}\mathrm{d}u^{\beta}\right)\boldsymbol{r}_1+$$

$$\left(\mathrm{d}b^2+\sum_{\alpha\beta=1}^{2}\Gamma_{\alpha\beta}^{2}b^{\alpha}\mathrm{d}u^{\beta}\right)\boldsymbol{r}_2$$

$$=\mathrm{D}(\boldsymbol{a})+\mathrm{D}(\boldsymbol{b});$$

(2) $\mathrm{D}(f\boldsymbol{a})$

$$=\left[\mathrm{d}(fa^1)+\sum_{\alpha\beta=1}^{2}\Gamma_{\alpha\beta}^{1}fa^{\alpha}\mathrm{d}u^{\beta}\right]\boldsymbol{r}_1+$$

$$\left[\mathrm{d}(fa^2)+\sum_{\alpha\beta=1}^{2}\Gamma_{\alpha\beta}^{2}fa^{\alpha}\mathrm{d}u^{\beta}\right]\boldsymbol{r}_2$$

$$=\left[\mathrm{d}f\cdot a^1+f\mathrm{d}a^1+f\sum_{\alpha\beta=1}^{2}\Gamma_{\alpha\beta}^{1}a^{\alpha}\mathrm{d}u^{\beta}\right]\boldsymbol{r}_1+$$

$$\left[\mathrm{d}f\cdot a^2+f\mathrm{d}a^2+f\sum_{\alpha\beta=1}^{2}\Gamma_{\alpha\beta}^{2}a^{\alpha}\mathrm{d}u^{\beta}\right]\boldsymbol{r}_2$$

$$=\mathrm{d}fa^1\boldsymbol{r}_1+\mathrm{d}fa^2\boldsymbol{r}_2+f\left(\mathrm{d}a^1+\sum_{\alpha\beta=1}^{2}\Gamma_{\alpha\beta}^{1}a^{\alpha}\mathrm{d}u^{\beta}\right)\boldsymbol{r}_1+$$

$$f\left(\mathrm{d}a^2+\sum_{\alpha\beta=1}^{2}\Gamma_{\alpha\beta}^{2}a^{\alpha}\mathrm{d}u^{\beta}\right)\boldsymbol{r}_2$$

$$=\mathrm{d}f\boldsymbol{a}+f\mathrm{D}\boldsymbol{a};$$

(3) $\mathrm{d}(\boldsymbol{a}\cdot\boldsymbol{b})=\mathrm{d}(a^1b^1+a^2b^2)$

$$=\mathrm{d}a^1\cdot b^1+a^1\cdot\mathrm{d}b^1+\mathrm{d}a^2\cdot b^2+a^2\cdot\mathrm{d}a^2,$$

$$\mathrm{D}\boldsymbol{a}\cdot\boldsymbol{b}+\boldsymbol{a}\cdot\mathrm{D}\boldsymbol{b}$$

$$=\left\{\left[\mathrm{d}a^1+\sum_{\alpha\beta=1}^{2}\Gamma_{\alpha\beta}^{1}a^{\alpha}\mathrm{d}u^{\beta}\right]\boldsymbol{r}_1+\left[\mathrm{d}a^2+\right.\right.$$

$$\sum_{\alpha\beta=1}^{2}\Gamma_{\alpha\beta}^{2}a^{\alpha}\mathrm{d}u^{\beta}\Big]\boldsymbol{r}_2\Big\}\cdot\boldsymbol{b}+\boldsymbol{a}\cdot\Big\{\Big[\mathrm{d}b^1+$$

$$\sum_{\alpha\beta=1}^{2}\Gamma_{\alpha\beta}^{1}b^{\alpha}\mathrm{d}u^{\beta}\Big]\boldsymbol{r}_1+\Big[\mathrm{d}b^2+\sum_{\alpha\beta=1}^{2}\Gamma_{\alpha\beta}^{2}b^{\alpha}\mathrm{d}u^{\beta}\Big]\boldsymbol{r}_2\Big\}$$

$$=\mathrm{d}a^1\cdot b^1+\mathrm{d}a^2\cdot b^2+\mathrm{d}b^1\cdot a^1+\mathrm{d}b^2\cdot a^2.$$

20 证明 设

$$\boldsymbol{a}=a^1\boldsymbol{r}_1+a^2\boldsymbol{r}_2$$

$$\boldsymbol{b}=b^1\boldsymbol{r}_1+b^2\boldsymbol{r}_2,$$

则

$$\boldsymbol{a}\cdot\boldsymbol{b}=a^1b^1+a^2b^2.$$

$$\frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{a}\cdot\boldsymbol{b})=\frac{\mathrm{d}a^1}{\mathrm{d}t}b^1+a^1\frac{\mathrm{d}b^1}{\mathrm{d}t}+a^2\frac{\mathrm{d}b^2}{\mathrm{d}t}+\frac{\mathrm{d}a^2}{\mathrm{d}t}b^2$$

$$=-\Big(\sum_{\alpha\beta}\Gamma_{\alpha\beta}^{1}a^{\alpha}\frac{\mathrm{d}u^{\beta}}{\mathrm{d}t}b^1+a^1\Gamma_{\alpha\beta}^{1}b^{\alpha}\frac{\mathrm{d}u^{\beta}}{\mathrm{d}t}+$$

$$a^2\sum_{\alpha\beta=1}^{2}\Gamma_{\alpha\beta}^{2}b^{\alpha}\frac{\mathrm{d}u^{\beta}}{\mathrm{d}t}+b^2\sum_{\alpha\beta=1}^{2}\Gamma_{\alpha\beta}^{2}a^{\alpha}\frac{\mathrm{d}u^{\beta}}{\mathrm{d}t}\Big)$$

$$=0.$$

所以，沿曲线平行移动时，$(\boldsymbol{a}\cdot\boldsymbol{b})=$常数.

又由于$(\boldsymbol{a}\cdot\boldsymbol{a})=$常数，所以沿曲线平行移动时，向量的长度不变，两向量夹角不变.